ÉTALE MOTIVES

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ABSTRACT. We define a theory of étale motives over a noetherian scheme. This provides a system of categories of complexes of motivic sheaves with integral coefficients which is closed under the six operations of Grothendieck. The rational part of these categories coincides with the triangulated categories of Beilinson motives (and is thus strongly related to algebraic K-theory). We extend the rigity theorem of Suslin and Voevodsky over a general base scheme. This can be reformulated by saying that torsion étale motives essentially coincide with the usual complexes of torsion étale sheaves (at least if we restrict ourselves to torsion prime to the residue characteristics). As a consequence, we obtain the expected results of absolute purity, of finiteness, and of Grothendieck duality for étale motives with integral coefficients, by putting together their counterparts for Beilinson motives and for torsion étale sheaves. Following Thomason's insights, this also provides a conceptual and convenient construction of the ℓ-adic realization of motives, as the homotopy ℓ-completion functor.

CONTENTS

INTRODUCTION

The aim of this article is to study various candidates for triangulated categories of étale motives. Already over a field, Voevodsky's triangulated category $\mathrm{DM}(k)$ comes with its étale counterpart $\mathrm{DM}_{\text{ét}}(k)$ (see [VSF00]). They coincide with $\mathbf{Q}$-coefficients, which means, for instance, that $\mathrm{DM}_{\text{ét}}(k,\mathbf{Q})$ can be used to understand algebraic K-theory up to torsion. On the other hand, as far as torsion coefficients are involved, the category $\mathrm{DM}_{\text{ét}}(k)$ is much closer to the topological world. Indeed, the rigidity theorem of Suslin and Voevodsky [SV96] means that for any positive integer n, prime to the characteristic of k, $\mathrm{DM}_{\text{ét}}(k,\mathbf{Z}/n\mathbf{Z})$ is equivalent to the derived category of $\mathbf{Z}/n\mathbf{Z}$-linear Galois modules. Over general base schemes, one expects to obtain the same pattern, and this is indeed what happens. We will use this repeatedly to prove properties of étale motives with integral coefficients: reduce to the case of rational coefficients, and then to the case of torsion coefficients (the latter being well understood since it belongs to the well established realm of étale cohomology). Then, there is the problem of the construction of such categories of étale motives with integral coefficients. There are several directions to do so.

One can consider the étale version of Morel and Voevodsky homotopy theory of schemes to produce and understand the triangulated category $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X,\mathbf{Z})$, obtained from complexes of sheaves of abelian groups on the smooth-étale site of X, by the usual $\mathbf{A}^1$-localisation and $\mathbf{P}^1$-stabilisation procedures. This is the right way, but not the easiest: this direction is studied by J. Ayoub in [Ayo], but with a little drawback: on has to work either with $\mathbf{Q}$-schemes, either with $\mathbf{Z}[1/2]$-coefficients. Although this restriction on 2-torsion should vanish once Morel's rigidty theorem (which is part of his program to prove the Friedlander-Milnor conjecture) will be established, this means that this is not an easy path.

Then, there are two other possibilities, which are the subject of this article. One can do as above, but taking the theory of étale sheaves with transfers, which defines a triangulated category $\mathrm{DM}_{\text{ét}}(X,\mathbf{Z})$. Or one can define another candidate, the category $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$, obtained from h-sheaves (we recall that the h-topology is the Grothenciedk topology on the category of noetherian schemes generated by étale coverings as well as by surjective proper maps). The category $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})$ is known to coincide with all the various notions of $\mathbf{Q}$-linear mixed motives which have the expected properties (mainly: expected relation with the graded piece of algebraic K-theory with respect to the γ-filtration, good behavior with respect to the six operations of Grothendieck). Up to a little variation, the first construction of triangulated categories of motives considered by Voevodsky was essentially the effective version of $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$; see [Voe96]. The category $\mathrm{DM}_{\text{ét}}(X,\mathbf{Q})$ has the disadvantage (for us) that we do not understand it enough, unless X is geometrically unibranch: in this case, we know that $\mathrm{DM}_{\text{ét}}(X,\mathbf{Q})$ and $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})$ coincide. In this article, we will see that $\mathrm{DM}_{\text{ét}}(X,R)$ and $\mathrm{DM}_{\mathrm{h}}(X,R)$ always coincide in the case of a ring of coefficients R of positive characteristic. We will also see that, if R is of characteristic $n > 0$ and if n is prime to the residue characteristics of X, then $\mathrm{DM}_{\text{ét}}(X,R)$ is canonically equivalent to the (unbounded) derived category $\mathrm{D}(X_{\text{ét}},R)$ of the category of sheaves of R-modules on the small étale site of X. This can be seen as generalisation of the rigidity theorem of Suslin and Voevodsky over a general base. From there, we will be able to see that the categories $\mathrm{DM}_{\mathrm{h}}(X,R)$ are well behaved with any coefficients, in the sense that the six operations act on them and preserve constructible objects; with mild assumptions on the base schemes, we will also obtain the existence of dualizing objects. This way of seeing torsion étale sheaves as motives gives a convenient way to produce ℓ-adic realisation functors, for any prime ℓ (odd or not).

As for the contents of this article, we will use the language we are the most familiar with: the one of [CD12]. A little recollection is given in the Appendix, in which one can find some complements about the notion of absolute purity and about the effect of the Artin-Schreier exact sequence in étale $\mathbf{A}^1$-homotopy theory.

The first part of this paper consists to formulate classical results of étale cohomology (such as the proper base change theorem, the smooth base change theorem, or cohomological descent) in terms of unbounded complexes. We also wanted to avoid any finiteness assumption about cohomological dimension, so that we have gathered what is needed to survive without such an hypothesis. These classical results are then used to study the triangulated categories $\mathrm{DM}_{\text{ét}}(X,R)$ for coefficients rings of positive characteristic, the crux being reached with the first version of the rigidity theorem: the comparison between $\mathrm{DM}_{\text{ét}}(X,R)$ and $\mathrm{D}(X_{\text{ét}},R)$. Beside classical properties of étale cohomology, the main point here is that, with this constraint on the coefficients, we prove the localization property for $\mathrm{DM}_{\text{ét}}(X,R)$ (which means that, if $Z\subset X$ is a closed subscheme with open complement U, then $\mathrm{DM}_{\text{ét}}(X,R)$ is obtained from $\mathrm{DM}_{\text{ét}}(U,R)$ and $\mathrm{DM}_{\text{ét}}(Z,R)$ by an adequate gluing procedure). This is a non trivial result (we do not know if this is true with rational coefficients). The second half of the paper is devoted to the study of the triangulated categories of h-motives $\mathrm{DM}_{\mathrm{h}}(X,R)$. We study at first the case of torsion coefficients, and see that we then get an equivalence with $\mathrm{DM}_{\text{ét}}(X,R)$ (the main argument for this being the proper descent theorem in étale cohomology extended to unbounded complexes, together with the relative rigidity theorem proved earlier). A significant part of our effort is then put in the yoga of reducing the proofs of properties of $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ to the case of $\mathbf{Q}$-coefficients and of $\mathbf{Z}/n\mathbf{Z}$-coefficients, so that we can gather what is known about $\mathbf{Q}$-linear mixed motives and classical torsion étale sheaves: we prove that the six operations preserve constructible objects in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ (for quasi-excellent noetherian schemes of finite dimension) and that there exists a dualizing motive in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ whenever X is separated and of finite type over a regular scheme S, itself of finite type over an excellent noetherian scheme of dimension ≤ 2. Finally, we describe ℓ-adic completion in terms of Bousfield localizations, in order to define suitable ℓ-adic realization functors.

Conventions

We will often fix a sub-category *Sch* of schemes and assume all the schemes are in *Sch*. Such an explicit category *Sch* will be fixed at the head of each section. When dealing with constructible objects (see below), we will also consider the subcategory Sch^{c} of *Sch* whose objects are the schemes in *Sch* which are moreover quasi-excellent and whose morphisms are the morphisms of finite presentation.

Unless stated otherwise, the word "smooth" ("étale") means smooth (étale) and separated of finite type. We will consider the following classes of morphisms in *Sch*:

- Ét for the class of étale morphisms,
- *Sm* for the class of smooth morphisms,
- $\mathscr{S}^{ft}$ for the class of morphisms of finite type.

Given a base scheme S, we let $X_{\text{ét}}$ (resp. Sm_S, $\mathscr{S}^{ft}_S$) be the sub-category of *Sch* made by S-schemes whose structural morphism is in Ét (resp. Sm, $\mathscr{S}^{ft}$).

Given any adjunction (F,G) of categories, we will denote generically by

$$ad(F,G): 1 \to GF \text{ resp. } ad'(F,G): FG \to 1$$

the respective unit and counit of the adjunction.

The letter R will often denote a ring of coefficients for the sheaves we consider.

1. Unbounded derived categories of étale sheaves

In this section we give a reminder of the properties of étale cohomology, as developped by Grothendieck and Artin in [AGV73]. There is nothing new, except some little complements about unbounded derived categories of étale sheaves.

1.1. Cohomological dimension.

1.1.1. Let X be a scheme. We denote by $X_{\text{ét}}$ the topos of sheaves on the small étale site of X. Given a ring R, we write $\mathrm{Sh}(X_{\text{ét}},R)$ for the category of sheaves of R-modules on $X_{\text{ét}}$. We will denote by $\mathrm{D}(X_{\text{ét}},R)$ the unbounded derived category of the abelian category $\mathrm{Sh}(X_{\text{ét}},R)$. Given an étale scheme U over X, we will write $R(U)$ for the sheaf representing evaluation at U, (i.e. the étale sheaf associated to the presheaf $R\langle \mathrm{Hom}_X(-,U)\rangle$).

Definition 1.1.2. A scheme X is of *finite étale cohomological dimension* there exists en integer n such that $H^i_{\text{ét}}(X,F)=0$ for any sheaf of abelian groups F over $X_{\text{ét}}$ and any integer $i>n$.

Let ℓ be a prime number.

A scheme X is of *finite ℓ-cohomological dimension* if there exists en integer n such that $H^i_{\text{ét}}(X,F)=0$ for any sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules F over $X_{\text{ét}}$ and any integer $i>n$. We denote by $\mathrm{cd}_\ell(X)$ the smallest integer n with the property above.

A field k is *of finite ℓ-cohomological dimension* if $\mathrm{Spec}(k)$ has this property.

Theorem 1.1.3 (Gabber)**.** *Let X be a strictly local noetherian scheme of dimension $d>0$, and ℓ a prime which is disinct of the residue characteristic of X. Then, for any open subscheme $U\subset X$, we have* $\mathrm{cd}_\ell(U)\leq 2d-1$.

For a proof, see [ILO12, Exposé XVIII].

Lemma 1.1.4. *Let X be a noetherian scheme of Krull dimension d. Then, for any sheaf of $\mathbf{Q}$-vector spaces F over $X_{\text{ét}}$, we have $H^i_{\text{ét}}(X,F)=0$ for $i>d$.*

Proof. Nisnevich cohomology and étale cohomology with coefficients in étale sheaves of $\mathbf{Q}$-vector spaces coincide, and Nisnevich cohomological dimension is bounded by the Krull dimension, which proves this assertion. □

Theorem 1.1.5 (Gabber). *Let S be a strictly local noetherian scheme and X an S-scheme of finite type. Then X is of finite étale cohomological dimension.*

Proof. An easy Mayer-Vietoris argument shows that it is sufficient to prove the theorem in the case where X is affine. For a point $x \in X$ with image $s \in S$, we write $d(x)$ for the degree of transcendence of the residue field $\kappa(x)$ over $\kappa(s)$. Note that, for any prime ℓ which is invertible in $\kappa(x)$, we have $\mathrm{cd}_\ell(\kappa(x)) \leq d(x)+\mathrm{cd}_\ell(\kappa(s))$; see [AGV73, Exposé X, Théorème 2.1]. Therefore, by virtue of Gabber's theorem 1.1.3, we have $\mathrm{cd}_\ell(\kappa(x)) \leq d(x)+2\dim(S)-1$. Let us define

$$N=\max\{1,\dim(X),\sup_{x\in X}(2\dim(S)-1+d(x)+2\mathrm{codim}(x))\}.$$

We will prove that $H^i_{\text{ét}}(X,F)=0$ for any sheaf F over $X_{\text{ét}}$ and any $i>N$. As X is quasi-compact and quasi-separated, the functors $H^i_{\text{ét}}(X,-)$ commute with filtered colimits; see [AGV73, Exposé VII, Proposition 3.3]. Therefore, we may assume that F is constructible; see [AGV73, Exposé IX, Corollaire 2.7.2]. We have an exact sequence of the form

$$0 \to T \to F \to C \to 0$$

where T is torsion and C is without torsion (in particular, C is flat over $\mathbf{Z}$). Therefore, we may assume that $F=T$ or $F=C$. We also have a short exact sequence

$$0 \to C \to C\otimes\mathbf{Q} \to C\otimes\mathbf{Q}/\mathbf{Z} \to 0$$

from which we deduce that

$$H^i_{\text{ét}}(X,C\otimes\mathbf{Q}/\mathbf{Z}) \simeq \varinjlim_n H^i_{\text{ét}}(X,C\otimes\mathbf{Z}/n\mathbf{Z})$$

for all i. Lemma 1.1.4 thus shows that it is sufficient to consider the case where F is the form T or $C\otimes\mathbf{Z}/n\mathbf{Z}$. But, as T is torsion and constructible, it is a $\mathbf{Z}/n\mathbf{Z}$-module for some integer $n\geq 1$. We are reduced to the case where F is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer $n \geq 1$. We can find a finite filtration

$$0=F_0\subset F_1\subset \ldots \subset F_k=F$$

such that F_{j+1}/F_j is a $\mathbf{Z}/\ell_j\mathbf{Z}$-module for any j, with ℓ_j a prime number: this follows from the fact such a filtration exists in the category of finite abelian groups, using [AGV73, Exposé IX, Proposition 2.14]. Therefore, we may assume that $n=\ell$ is a prime number.

We will prove that, for any sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules F over $X_{\text{ét}}$, we have $H^i_{\text{ét}}(X,F)=0$ for $i>N$. Let $Z=\mathrm{Spec}(\mathbf{Z}/\ell\mathbf{Z})\times X$ and $U=X-Z$. We have a closed immersion $i: Z\to X$ and its open complement $j: U\to X$, which gives a long exact sequence

$$H^i_{\text{ét}}(Z,i^!(F)) \to H^i_{\text{ét}}(X,F) \to H^i_{\text{ét}}(U,j^*(F)) \to H^{i+1}_{\text{ét}}(Z,i^!(F)).$$

By virtue of [AGV73, Exposé X, Théorème 5.1], we have $H^i_{\text{ét}}(X,i^!(F))=0$ for $i>1$. On the other hand, we have

$$H^i_{\text{ét}}(U,j^*(F))=0 \text{ for any integer } i\leq \sup_{x\in U}(\mathrm{cd}_\ell(k(x))+2\mathrm{codim}(x))$$

(see [ILO12, Exposé XVIII, Lemma 2.2]). □

Remark 1.1.6. Gabber also proved the Affine Lefschetz Theorem: if X is an excellent strictly local scheme of dimension d, for any open subscheme $U \subset X$, we have $\mathrm{cd}_\ell(U) \leq d$; see [ILO12, Exposé XV, Corollaire 1.2.2]. In the case of excellent schemes of characteristic zero, this had been proved by Artin, using Hironaka's resolution of singularities; see [AGV73, Exposé XIX, Corollaire 6.3]. The case of a scheme of finite type over an excellent scheme of dimension ≤ 1 was also known (this follows easily from [AGV73, Exposé X, Proposition 3.2]).

Lemma 1.1.7. *Let $\mathscr{A}$ be an abelian Grothendieck category. We also consider a right exact functor*

$$F : \mathscr{A} \to \mathbf{Z}\text{-Mod}\,,$$

and we denote by

$$\mathbf{R}F : \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathbf{Z}\text{-Mod})$$

its total right derived functor. We suppose that the functor

$$\mathscr{A} \to \mathbf{Z}\text{-Mod}\,, \quad A \mapsto \mathbf{R}^n F(A)$$

commutes with small filtered colimits for any integer $n \geq 0$.

Then, the following conditions are equivalent.

(i) *The functor*

$$\mathrm{C}(\mathscr{A}) \to \mathbf{Z}\text{-Mod}\,, \quad K \mapsto H^0\mathbf{R}F(K)$$

commutes with small filtered colimits.

(ii) *The functor $\mathbf{R}F$ commutes with small sums.*

(iii) *The functor $\mathbf{R}F$ commutes with countable sums.*

(iv) *For any degreewise F-acyclic complexe K, the natural map $F(K) \to \mathbf{R}F(K)$ is an isomorphism in $\mathrm{D}(\mathbf{Z}\text{-Mod})$.*

Moreover, the four conditions above are verified whenever the functor F is of finite cohomological dimension.

Proof. It is clear that (i)⇒(ii)⇒(iii). It is also easy to see that property (iv) implies property (i). Indeed, our assumption on F implies that the class of F-acyclic objects is closed under filtered colimits, which implies that the class of degreewise F-acyclic complexes has the same property. On the other hand, property (iv) implies that the functor $\mathbf{R}F$ may constructed using resolutions by degreewise F-acyclic complexes, from which property (i) follows immediately.

Let us show that condition (iii) implies condition (iv). Consider a sequence of morphisms of complexes of $\mathscr{A}$:

$$K_0 \to K_1 \to \cdots \to K_n \to K_{n+1} \to \dots \quad , \qquad n \geq 0\,.$$

We then have a map

$$1 - d : \bigoplus_n K_n \to \bigoplus K_n\,,$$

where d is the morphism induced by the maps $K_n \to K_{n+1}$. The cone of $1-d$ (the cokernel of $1-d$, respectively) is the homotopy colimit (the colimit, respectively) of the diagram $\{K_n\}$. Moreover, as filtered colimits are exact in $\mathscr{A}$, the canonical map

$$\mathbf{L}\varinjlim_n K_n \to \varinjlim_n K$$

is an isomorphism in $\mathrm{D}(\mathscr{A})$. As a consequence, it follows from condition (iii) that, if K belongs to $\mathrm{C}(\mathscr{A})$, we have a natural long exact sequence of shape

$$\cdots \to \bigoplus_n H^i \mathbf{R}F(K_n) \xrightarrow{1-d} \bigoplus_n H^i \mathbf{R}F(K_n) \to H^i \mathbf{R}F(\varinjlim_n K_n) \to \cdots$$

It is easy to deduce from this that, assuming condition (iii), the natural map

$$\varinjlim_n H^0 \mathbf{R}F(K_n) \to H^0 \mathbf{R}F(\varinjlim_n K_n)$$

is always invertible.

For an integer n, let us write $\sigma^{\geq n}(K)$ for the 'troncation bête', defined as $\sigma^{\geq n}(K)^i = K^i$ if $i \geq n$ and $\sigma^{\geq n}(K)^i = 0$ otherwise. We can then write

$$\varinjlim_n \sigma^{\geq m}(K) \simeq K\,.$$

Suppose furthermore that the complex K is degreewise F-acyclic. Then $\sigma^{\geq n}(K)$ has the same property and has moreover the good taste of being bounded below. Therefore, the map

$$F(\sigma^{\geq n}(K)) \to \mathbf{R}F(\sigma^{\geq n}(K))$$

is an isomorphism for any integer n. As both the functors H^0F and $H^0\mathbf{R}F$ commutes with $\varinjlim_n$, we conclude that property (iv) is verified.

The fact that property (iv) is true whenever F is of finite cohomological dimension is well known (it is already in the book of Cartan and Eilenberg in the case where $\mathscr{A}$ is a category of modules over some ring, and a general argument may be found for instance in [SV00a, Lemma 0.4.1]). □

1.1.8. Given a topos T and a ring R, we will write $\mathrm{Sh}(T,R)$ for the category of R-modules in T (or, equivalently, the category of sheaves of R-modules over T). If $\mathscr{G}$ is a generating family of T, the category $\mathrm{C}(\mathrm{Sh}(X,R))$ is endowed with the *projective model category structure* with respect to $\mathscr{G}$ (see [CD09, Example 2.3, Theorem 2.5, Corollary 5.5]): the weak equivalences are the quasi-isomorphisms, while the fibrant objects are the complexes of sheaves of R-modules K such that, for any object U in $\mathscr{G}$, the natural map

$$H^n(\Gamma(U,K)) \to H^n(U,K)$$

is an isomorphism for any integer n (where $H^n(U,K)$ denotes the hypercohomology groups of U with coefficients in K). The fibrations (trivial fibrations) are the morphisms of shape $p : K \to L$ with the following properties:

(i) for any object U in $\mathscr{G}$, the map $p : \Gamma(U,K) \to \Gamma(U,L)$ is degreewise surjective;
(ii) the kernel of p is fibrant (the complex $\Gamma(U,\ker(p))$ is acyclic for any U in $\mathscr{G}$, respectively).

Moreover, for any object U in $\mathscr{G}$, the object $R(U)$ (the free sheaf of R-modules generated by U), seen as a complex concentrated in degree zero, is cofibrant. We will write $\mathrm{D}(T,R)$ for the (unbounded) derived category of $\mathrm{Sh}(T,R)$.

If a topos T is canonically constructed as the category of sheaves on a Grothendieck site, the class of representable sheaves is a generating family of T, and, unless we explicitely specify another choice, the projective model structures on the categories of sheaves of R-modules over T will be considered with respect this generating family. For instance, for a scheme X, we will always understand the topos $X_{\text{ét}}$ as the category of sheaves over the small étale site of X, so that its canonical generating family is given by the collection of all étale schemes of finite presentation over X.

Proposition 1.1.9. *Consider a topos T and a ring R. We suppose that T is endowed with a generating family $\mathcal{G}$ such that any $U \in \mathcal{G}$ is is coherent and of finite cohomological dimension for R-linear coefficients. Then, for any $U \in \mathcal{G}$, the functor*

$$\mathrm{C}(\mathrm{Sh}(T,R)) \to R\text{-Mod} \quad , \qquad K \mapsto \mathrm{Hom}_{\mathrm{D}(T,R)}(R(U),K) = H^0(U,K)$$

preserves small filtered colimits.

In particular, the family $\{R(U) \mid U \in \mathcal{G}\}$ form a family of compact generators of the triangulated category $\mathrm{D}(T,R)$.

Proof. This is a direct consequence of Lemma 1.1.7. □

Lemma 1.1.10. *Let T be a topos and U a coherent object of T. For any sheaf of abelian groups F over T, the natural map*

$$H^i(U,F) \otimes \mathbf{Q} \to H^i(U,F \otimes \mathbf{Q})$$

is invertible for any integer i. In particular, tensoring with $\mathbf{Q}$ preserves $\Gamma(U,-)$-acyclic sheaves over T. If moreover U is of finite cohomological dimension with rational coefficients, then, for any complex of sheaves of abelian groups K over T, the canonical map

$$H^i(U,K) \otimes \mathbf{Q} \to H^i(U,K \otimes \mathbf{Q})$$

is bijective for any integer i.

Proof. The first assertion immediately follows from the fact that the functor $H^i(U,-)$ preserves filtering colimits of sheaves. The second assertion is an immediate consequence of the first. Finally, the last assertion is a direct consequence of Lemma 1.1.7. □

Proposition 1.1.11. *Let X be a noetherian scheme of finite dimension. For any complex of étale sheaves of $\mathbf{Q}$-vector spaces K, the natural map*

$$H^i_{\text{ét}}(X,K) \otimes \mathbf{Q} \to H^i_{\text{ét}}(X,K \otimes \mathbf{Q})$$

is bijective for any integer i.

Proof. By virtue of Lemma 1.1.4, this obviously is a particular case of the preceding lemma. □

The following lemma is the main tool to extend results about unbounded complexes of sheaves which are known under a global finite cohomological dimension hypothesis to contexts where finite cohomological dimension is only assumed pointwise (in the topos theoretic sense). This will be used to extend to unbounded complexes of étale sheaves the smooth base change formula as well as the proper cohomological descent theorem. We will freely use the language and the results of [AGV73, Exposé VII] about coherent topoi and filtering limits of these.

Lemma 1.1.12. *Consider a ring of coefficients R and an essentially small cofiltering category I as well as a fibred topos $S \to I$. For each index i we consider a given generating family $\mathcal{G}_i$ of the topos S_i. We write $T = \varprojlim_I S$ for the limit topos, and $\pi_i : T \to S_i$ for the canonical projections. We then have a canonical generating family $\mathcal{G}$ of T, which consists of objects of the form $\pi_i^*(X_i)$, where X_i is an element of the class $\mathcal{G}_i$. Given a map $f : i \to j$ in I and a sheaf F_j over S_j, we will write F_i for the sheaf over S_i obtained by applying the pullback functor $f^* : S_j \to S_i$ to F_j. We will assume that the following properties are satisfied:*

(i) *For each index i, any object in $\mathscr{G}_i$ is coherent (in particular, the topos S_i is coherent).*
(ii) *For any map $f: i \to j$ in I, the corresponding pullback functor $f^*: S_j \to S_i$ sends any object in $\mathscr{G}_j$ to an object isomorphic to an element of $\mathscr{G}_i$ (in particular, the morphism of topoi $S_i \to S_j$ is coherent).*
(iii) *For any map $f: i \to j$ in I, the pullback functor $f^*: S_j \to S_i$ has a left adjoint $f_\sharp: S_i \to S_j$ which sends any object in $\mathscr{G}_i$ to an object isomorphic to an element of $\mathscr{G}_j$.*
(iv) *Any object in $\mathscr{G}$, has finite cohomological dimension with respect to sheaf cohomology of R-modules.*

Then, for any index i_0, the pullback functor $\pi^*_{i_0}: \mathrm{C}(\mathrm{Sh}(S_{i_0},R)) \to \mathrm{C}(\mathrm{Sh}(T,R))$ preserves the fibrations of the projective model structures. Moreover, for any object U_{i_0} of $\mathscr{G}_{i_0}$, and for any complex K_{i_0} of $\mathrm{Sh}(S_{i_0},R)$, if $U = \pi^*_{i_0}(U_{i_0})$ and $K = \pi^*_{i_0}(K_{i_0})$, then the canonical map

$$\varinjlim_{i \to i_0} H^n(U_i,K_i) \to H^n(U,K) \tag{1.1.12.a}$$

is bijective for any integer n.

Proof. Note that formula (1.1.12.a) is known to hold whenever K_{i_0} is concentrated in degree zero and $n = 0$; see [AGV73, Exposé VII, Corollaire 8.5.7]. This shows that condition (i) of 1.1.8 is preserved by the functor $\pi^*_{i_0}$. Therefore, in order to prove that the functor $\pi^*_{i_0}$ preserves fibrations, it is sufficient to prove that it preserves fibrant objects. Let K_{i_0} be a fibrant object of $\mathrm{C}(\mathrm{Sh}(S_{i_0},R))$. We have to prove that the natural map

$$H^n(\Gamma(U,K)) \to H^n(U,K) \tag{1.1.12.b}$$

is an isomorphism for any object U in $\mathscr{G}$. For any map $f: i \to j$ in I, condition (iii) above implies that the functor f^* preserves fibrations as well as trivial fibrations (whence it preserves fibrant objects as well). Possibly up to the replacement of i_0 by some other index above it, we may assume that U is the pullback of an object U_{i_0} in $\mathscr{G}_{i_0}$. Formula (1.1.12.a) in the case of complexes concentrated in degree zero then gives us a canonical isomorphism

$$H^n(\Gamma(U,K)) \simeq \varinjlim_{i \to i_0} H^n(\Gamma(U_i,K_i)). \tag{1.1.12.c}$$

As K_i is fibrant for any map $i \to i_0$, we thus get a natural identification

$$H^n(\Gamma(U,K)) \simeq \varinjlim_{i \to i_0} H^n(U_i,K_i). \tag{1.1.12.d}$$

In other words, we must prove that the natural map (1.1.12.a) is invertible for any (fibrant) unbounded complex of sheaves K_{i_0} and any object U_{i_0} in $\mathscr{G}_{i_0}$.

For this purpose, we will work with the *injective model category structure* on $\mathrm{C}(\mathrm{Sh}(S_{i_0},R))$ (see [CD09, 2.1]), whose weak equivalences are the quasi-isomorphisms, and whose cofibrations are the monomorphisms: as any object of a model category has a fibrant resolution, it is sufficient to prove that (1.1.12.a) is invertible whenever K_{i_0} is fibrant for the injective model structure. In this case, the complex K_{i_0} is degreewise an injective object of $\mathrm{Sh}(S_{i_0},R)$. This implies that its image by the functor $\pi^*_{i_0}$ is a complex of $\Gamma(U,-)$-acyclic sheaves; see [AGV73, Exposé VII, Lemme

8.7.2]. Therefore, using Lemma 1.1.7 and assumption (iv), the map (1.1.12.b) is invertible for such a complex K, from which we immediately deduce that (1.1.12.a) is invertible. □

Remark 1.1.13. With the same assumptions as in the preceding lemma, in the case $R = \mathbf{Q}$, for any complex of sheaves of abelian groups K_{i_0} over S_{i_0} and any object U_{i_0} in $\mathscr{G}_{i_0}$, the natural maps

$$\varinjlim_{i \to i_0} H^n(U_i, K_i) \otimes \mathbf{Q} \to H^n(U, K \otimes \mathbf{Q})$$

are isomorphims. Indeed, we know from Lemma 1.1.10 that tensoring with $\mathbf{Q}$ preserves $\Gamma(U,-)$-acyclic sheaves of abelian groups over T for any object U in $\mathscr{G}$. Therefore, as we may assume that K_{i_0} is fibrant for the injective model structure, which implies, by [AGV73, Exposé VII, Lemme 8.7.2], that K is degreewise $\Gamma(U,-)$-acyclic, the complex $K \otimes \mathbf{Q}$ has the same property. As the functors $\Gamma(V,-)$ commute with $(-) \otimes \mathbf{Q}$ for any coherent sheaf of sets V, we conclude as in the proof of the preceding lemma.

Theorem 1.1.14. *Consider a cartesian square of locally noetherian schemes*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with the following properties.

(a) *The scheme S' is the limit of a projective system of étale quasi-compact and quasi-separated schemes over S, with affine transition morphisms.*
(b) *The morphism f is of finite type.*

Then, for any object K of $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$*, the base change map*

$$g^* \mathbf{R}f_*(K) \to \mathbf{R}f'_* h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}}, \mathbf{Z})$.

Proof. Let us first prove the theorem under the additional assumption that the scheme S' is strictly local. By virtue of Theorem 1.1.5, any scheme of finite type over S' is of finite étale cohomological dimension. If $S' = \varprojlim_i S_i$, where $\{S_i\}$ is a projective system of étale S-schemes with affine transition maps, then the topos $S'_{\text{ét}}$ is canonically equivalent to the projective limit of topoi $\varprojlim_i S_{i,\text{ét}}$; see [AGV73, Exposé VII, Theorem 5.7]. Similarly, if we write $X_i = S_i \times_S X$, we have $X' \simeq \varprojlim_i X_i$ and $X' \simeq \varprojlim_i X_{i,\text{ét}}$. Note that, for any étale map $u : T' \to T$, the pullback functor $u^* : T_{\text{ét}} \to T'_{\text{ét}}$ has a left adjoint (because the category $T'_{\text{ét}}$ is naturally equivalent to the category $T_{\text{ét}}/T'$, where T' is seen as a sheaf over $T_{\text{ét}}$), and that any map between étale schemes is itself étale, from which one deduces that condition (iii) of Lemma 1.1.12 is satisfied for both projective systems $\{S_i\}$ and $\{X_i\}$. As the other assumptions of this lemma are also verified, we see that the functors g^* and h^* preserve finite limits, weak equivalences, as well as fibrations of the projective model structures. On the other hand, the functors f_* and f'_* are always right Quillen functors for the projective model structures. We deduce from this that we have natural isomorphism as the level of total right derived functors:

$$\mathbf{R}(g^* f_*) \simeq \mathbf{R}g^* \mathbf{R}f_* = g^* \mathbf{R}f_* \text{ and } \mathbf{R}(f'_* h^*) \simeq \mathbf{R}f'_* \mathbf{R}h^* = \mathbf{R}f'_* h^* .$$

As the natural map $g^* f_*(F) \to f'_* h^*(F)$ is an isomorphism for any sheaf F over $X_{\text{ét}}$ (one checks this by first replacing S' by each of the S_i's and X' by the X_i's, and then proceed to the limit), this proves that, under our additional assumptions, the natural transformation $g^* \mathbf{R}f_* \to \mathbf{R}f'_* h^*$ is invertible.

The general case can now be proven as follows. It is sufficient to prove that, for any geometric point ξ' of S', if S'' denotes the spectrum of the strict henselisation of the local ring $\mathscr{O}_{S',\xi'}$, and if $g' : S'' \to S'$ is the natural map, then the morphism

$$g'^* g^* \mathbf{R}f_*(K) \to g'^* \mathbf{R}f'_* h^*(K)$$

is invertible for any object K of $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$. We then have the following pullback squares

$$\begin{tikzcd}
X'' \arrow[r, "h'"] \arrow[d, "f''"'] & X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S'' \arrow[r, "g'"] & S' \arrow[r, "g"] & S\,.
\end{tikzcd}$$

Therefore, applying twice the first part of this proof, we obtain two canonical isomorphisms

$$g'^* \mathbf{R}f'_* h^*(K) \to \mathbf{R}f''_* h'^* h^*(K) \text{ and } g'^* g^* \mathbf{R}f_*(K) \to \mathbf{R}f''_* h'^* h^*(K)\,.$$

As we have a commutative triangle

$$\begin{tikzcd}
g'^* g^* \mathbf{R}f_*(K) \arrow[rr] \arrow[dr, "\simeq"'] & & g'^* \mathbf{R}f'_* h^*(K) \arrow[dl, "\simeq"] \\
& \mathbf{R}f''_* h'^* h^*(K) &
\end{tikzcd},$$

this shows that the map $g^* \mathbf{R}f_*(K) \to \mathbf{R}f'_* h^*(K)$ is invertible. □

Corollary 1.1.15. *Let $f : X \to S$ be a morphism of finite type between locally noetherian schemes. The induced derived direct image functor*

$$\mathbf{R}f_* : \mathrm{D}(X_{\text{ét}}, \mathbf{Z}) \to \mathrm{D}(S_{\text{ét}}, \mathbf{Z})$$

preserves small sums.

Proof. By virtue of the preceding theorem, we may assume that S is strictly local. We then know from Theorem 1.1.5 and Proposition 1.1.9, that both $\mathrm{D}(S_{\text{ét}}, \mathbf{Z})$ and $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$ are compactly generated triangulated categories (with canonical families of compact generators given by sheaves of shape $\mathbf{Z}(U)$ for U étale over the base), and that the functor $f^* : \mathrm{D}(S_{\text{ét}}, \mathbf{Z}) \to \mathrm{D}(X_{\text{ét}}, \mathbf{Z})$ preserves compact objects. This immediately implies that its right adjoint of $\mathbf{R}f_*$ commutes with small sums. □

1.2. **Proper base change isomorphism.**

Theorem 1.2.1. *Consider a cartesian square of schemes*

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

with f proper. Then, for any ring R of positive characteristic, and for any object K of $\mathrm{D}(X_{\text{ét}},R)$, the canonical map

$$g^*\,\mathbf{R}f_*(K)\to\mathbf{R}f'_*\,h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}},R)$.

Corollary 1.2.2. *Let $f: X\to S$ be a proper morphism of schemes, and let ξ be a geometric point of S. Let us denote by X_ξ the fiber of X over ξ. Then, for any ring R of positive characteristic, and for any object K of $\mathrm{D}(X_{\text{ét}},R)$, the natural map*

$$\mathbf{R}f_*(K)_\xi\to\mathbf{R}\Gamma(X_\xi,K|_{X_\xi})$$

is an isomorphism in the derived category of the category of R-modules.

Let us see that Corollary 1.2.2 implies Theorem 1.2.1.

In order to prove that the map $g^*\,\mathbf{R}f_*(K)\to\mathbf{R}f'_*\,h^*(K)$ is invertible, it is sufficient to prove that, for any geometric point ξ' of S', if we write $\xi=g(\xi')$, the induced map

$$(g^*\,\mathbf{R}f_*(K))_{\xi'}=\mathbf{R}f_*(K)_\xi\to\mathbf{R}f'_*(h^*(K))_{\xi'}$$

is an isomorphism. If X_ξ and $X'_{\xi'}$ denote the fiber of X over ξ and of X' over ξ' respectively, as the commutative square of Theorem 1.2.1 is cartesian, the natural map $X'_{\xi'}\to X_\xi$ is an isomorphism. Moreover, applying twice Corollary 1.2.2 gives canonical isomorphisms

$$\mathbf{R}f_*(K)_\xi\simeq\mathbf{R}\Gamma(X_\xi,K|_{X_\xi})\quad\text{and}\quad\mathbf{R}f'_*(h^*(K))_{\xi'}\simeq\mathbf{R}\Gamma(X'_{\xi'},h^*(K)|_{X'_{\xi'}}).$$

As the square

$$\begin{tikzcd}
\mathbf{R}f_*(K)_\xi \arrow[r] \arrow[d, "\wr"] & \mathbf{R}f'_*(h^*(K))_{\xi'} \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(X_\xi,K|_{X_\xi}) \arrow[r, "\sim"] & \mathbf{R}\Gamma(X'_{\xi'},h^*(K)|_{X'_{\xi'}})
\end{tikzcd}$$

commutes, this proves the theorem.

Proof of Corollary 1.2.2. By virtue of [AGV73, Exposé XII, Corollaire 5.2], we already know this corollary is true whenever K is actually a sheaf of R-modules over $X_{\text{ét}}$, from which we easily deduce that this is an isomorphism for K a bounded complex of sheaves of R-modules. Note that X_ξ is of finite cohomological dimension (by Theorem 1.1.5, although this is here much more elementary, as this readily follows from [AGV73, Exposé X, 4.3 and 5.2]). Moreover, as the fiber functor

$$\mathrm{Sh}(S_{\text{ét}},R)\to R\text{-Mod}\,,\quad F\mapsto F_\xi$$

is exact, the functor $K\mapsto\mathbf{R}f_*(K)_\xi$ is the total right derived functor of the left exact functor $F\mapsto f_*(F)_\xi\simeq\Gamma(X_\xi,F|_{X_\xi})$, which is thus of finite cohomological dimension; see [AGV73, Exposé XII, 5.2 and 5.3]. Therefore, by virtue of Lemma 1.1.7, the map $H^i(\mathbf{R}f_*(K)_\xi)\to H^i_{\text{ét}}(X_\xi,K|_{X_\xi})$ is a natural transformation between functors which preserve small filtering colimits of complexes of sheaves. As any complex is a filtered colimit of bounded complexes, this ends the proof. $\square$

Corollary 1.2.3. *For any proper morphism $f: X\to S$, and for any ring R of positive characteristic, the functor*

$$\mathbf{R}f_*:\mathrm{D}(X_{\text{ét}},R)\to\mathrm{D}(S_{\text{ét}},R)$$

has a right adjoint

$$f^! : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(X_{\text{ét}}, R).$$

Proof. By virtue of the Brown representability theorem, it is sufficient to prove that $\mathbf{R}f_*$ preserves small sums. For this purpose, it is sufficient to prove that, for any geometric point ξ of S, the functor $\mathbf{R}f_*(-)_\xi : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{D}(R\text{-Mod})$ preserves small sums. This readily follows from Corollaries 1.2.2 and 1.1.15. □

1.3. **Smooth base change isomorphism and homotopy invariance.**

Theorem 1.3.1. *Consider the cartesian square of locally noetherian schemes below, with g a smooth morphism, and f of finite type.*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Consider a ring R of positive characteristic which is prime to the residue characteristics of S. Then, for any object K of $\mathrm{D}(X_{\text{ét}}, R)$, the map

$$g^* \mathbf{R}f_*(K) \to \mathbf{R}f'_* h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}}, R)$.

Proof. The smallest triangulated full subcategory of $\mathrm{D}(X_{\text{ét}}, R)$ which is closed under small sums, and which contains sheaves of R-modules over $X_{\text{ét}}$, is the whole category $\mathrm{D}(X_{\text{ét}}, R)$. Therefore, by virtue of Corollary 1.1.15, it is sufficient to prove that, for any sheaf of R-modules F over $X_{\text{ét}}$, the map

$$g^* \mathbf{R}f_*(F) \to \mathbf{R}f'_* h^*(F)$$

is an isomorphism. This follows from [AGV73, Exposé XVI, Corollaire 1.2]. □

Theorem 1.3.2. *Let S be a locally noetherian scheme and $p : V \to S$ be a vector bundle. Consider a ring R of positive characteristic which is prime to the residue characteristics of S. Then the pullback functor $p^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(V_{\text{ét}}, R)$ is fully faithful.*

Proof. The property that p^* is fully faithful is local over S for the Zariski topology, so that may assume that $V = \mathbf{A}^n_S$, and even that $n = 1$. We have to check that, for any complex K of sheaves of R-modules over $S_{\text{ét}}$, the unit map $K \to \mathbf{R}p_* p^*(K)$ is an isomorphism in $\mathrm{D}(S_{\text{ét}}, R)$. By Corollary 1.1.15, the functor $\mathbf{R}p_*$ preserves small sums, so that we may assume that K is concentrated in degree zero (by the same argument as in the preceding proof). This follows then from [AGV73, Exposé XV, Corollaire 2.2]. □

2. THE PREMOTIVIC ÉTALE CATEGORY

In this section, R can be any ring, while the schemes will be noetherian. Unless stated otherwise, given any base scheme S, S-schemes are assumed to be separated and of finite type.

The category of separated smooth S-schemes of finite type Sm_S, endowed with the étale topology, is called the *smooth-étale site*. We denote by $\mathrm{Sh}_{\text{ét}}(S, R)$ the category of sheaves of R-modules on this site (this has to be distinguished from the category of sheaves on the small site; see 1.1.1).

2.1. **Étale sheaves with transfers.**

2.1.1. We recall here the theory of finite correspondences and of sheaves with transfers introduced by Suslin and Voevodsky [SV00b]. The precise definitions and conventions can be found in [CD12, section 9].

Let us fix a sub-ring Λ of $\mathbf{Q}$ as the ring of coefficients of all cycles considered in this paragraph. Given any S-scheme X, we denote by

$$c_0(X/S)_\Lambda$$

the abelian group of cycles α in X with coefficients in Λ such that α is finite and Λ-universal over S (*ie* the support of α is finite over S and α/S satisfies the definition [CD12, 9.1.1]).

Given any S-schemes X and Y, we put

$$c_S(X,Y)_\Lambda := c_0(X \times_S Y/X)_\Lambda$$

and call its elements the *finite S-correspondences from X to Y* (cf. [CD12, 9.1.2]).

These correspondences can be composed and we denote by $Sm^{cor}_{\Lambda,S}$ the category whose objects are smooth S-schemes and morphisms are finite S-correspondences (see [CD12, 9.1.8] for $\mathscr{P}$ the class of smooth separated morphisms of finite type).

We can define a functor

$$\gamma_S : Sm_S \to Sm^{cor}_{\Lambda,S} \tag{2.1.1.a}$$

which is the identity on objects and associates to an S-morphism its graph seen as a finite S-correspondence [CD12, 9.1.8.1].

When the coefficients are not indicated in the notation, it is understood that $\Lambda = \mathbf{Z}$. This will always be the case in the rest of this section.

Definition 2.1.2. (see [CD12, 10.1.1 and 10.2.1]) An *R-presheaf with transfers over S* is an additive presheaf of R-modules on Sm^{cor}_S. We denote by $\mathrm{PSh}^{tr}(S,R)$ the corresponding category.

An *étale R-sheaf with transfers over S* is an R-presheaf with transfers F such that $F \circ \gamma_S$ is a sheaf for the étale topology. We denote by $\mathrm{Sh}^{tr}_{\text{ét}}(S,R)$ the corresponding full subcategory of $\mathrm{PSh}^{tr}(S,R)$.

Thus, by definition, we have an obvious functor:

$$\gamma_* : \mathrm{Sh}^{tr}_{\text{ét}}(S,R) \to \mathrm{Sh}_{\text{ét}}(S,R), F \mapsto F \circ \gamma. \tag{2.1.2.a}$$

2.1.3. Given any S-scheme X, we let $R^{tr}_S(X)$ be the following R-presheaf with transfers:

$$Y \mapsto c_S(Y,X) \otimes_{\mathbf{Z}} R.$$

Proposition 2.1.4. *The presheaf $R^{tr}_S(X)$ is an étale R-sheaf with transfers.*

Proof. In the case where $R = \mathbf{Z}$ this is [CD12, Proposition 10.2.4]. For the general case, we observe that for any smooth S-scheme Y, $c_S(Y,X)$ is a free abelian group. Indeed, it is a sub-$\mathbf{Z}$-module of the free $\mathbf{Z}$-module of cycles in $Y \times_S X$. Thus, we have

$$\mathrm{Tor}^1_{\mathbf{Z}}(c_S(Y,X), R) = 0, \tag{2.1.4.a}$$

and the general case follows from the case $R = \mathbf{Z}$. $\square$

2.1.5. Let $Y_\bullet$ be a simplicial S-scheme. If we apply R_S^{tr} pointwise, we obtain a simplicial object of the additive category $\mathrm{Sh}_{\text{ét}}^{tr}(S,R)$. We denote by $R_S^{tr}(Y_\bullet)$ the complex associated with this simplicial object. This is obviously functorial in $Y_\bullet$.

The following proposition is the main technical point of this section.

Proposition 2.1.6. *Let $p : Y_\bullet \to X$ be an étale hypercover of X in the category of S-schemes. Then the induced map*

$$p_* : \gamma_* R_S^{tr}(Y_\bullet) \to \gamma_* R_S^{tr}(X)$$

is a quasi-isomorphism of complexes of étale R-sheaves.

Proof. The general case follows from the case $R = \mathbf{Z}$ – using the argument (2.1.4.a). In the proof, a *geometric point* will mean a point with coefficients in an algebraically closed field – not only separably closed[1]. We will use the abelian group $c_0(Z/S)$ defined for any S-scheme Z in 2.1.1. Remember that it is covariantly functorial in Z; see [CD12, 9.1.1].

First step. We reduce to the case where S is strictly local and to prove that the canonical map of complexes of $\mathbf{Z}$-modules

$$p_* : c_0(Y_\bullet/S) \to c_0(X/S) \tag{2.1.6.a}$$

is a quasi-isomorphism.

Indeed, to check that p_* is an isomorphism, it is sufficient to look at fibers over a point of the smooth-étale site. Such a point corresponds to a smooth S-scheme T with a geometric point $\bar{t}$; we have to show that the map of complexes of abelian groups:

$$\varinjlim_{V \in \mathcal{V}_{\bar{t}}(T)} c_S(V, Y_\bullet) \to \varinjlim_{V \in \mathcal{V}_{\bar{t}}(T)} c_S(V, X)$$

is an isomorphism.

Let T_0 be the strict local scheme of T at $\bar{t}$. By virtue of [CD12, 8.3.9], for any smooth S-scheme W, the canonical map:

$$\varinjlim_{V \in \mathcal{V}_{\bar{t}}(T)} c_S(V, W) \to c_0(Z \times_S T_0/T_0) = c_{T_0}(T_0, W \times_S T_0).$$

is an isomorphism. This concludes the first step as we may replace S by T_0 as well as p by $p \times_S T_0$.

Second step. We reduce to prove that (2.1.6.a) is a quasi-isomorphism in the case where X is connected and finite over S.

Let $\mathcal{Z}$ be the set of closed subschemes Z of X which are finite over S, ordered by inclusion. Given such a Z, we consider the canonical immersion $i : Z \to X$ and the pullback square:

$$\begin{tikzcd} Z \times_S Y_\bullet \arrow[r, "p_Z"] \arrow[d, "k"'] & Z \arrow[d, "i"] \\ Y_\bullet \arrow[r, "p"] & X. \end{tikzcd}$$

[1] In the proof, we will only use the fact that *any* surjective family of geometric points on a scheme X gives a conservative family of points of the small étale site of X; see [AGV73, VIII, 3.5].

We thus obtain a commutative diagram:

$$\begin{tikzcd} c_0(Z \times_X Y_\bullet/S) \arrow[r, "p_{Z*}"] \arrow[d] & c_0(Z/S) \arrow[d] \\ c_0(Y_\bullet/S) \arrow[r, "p_*"] & c_0(X/S). \end{tikzcd}$$

In this diagram, the vertical maps are injective and we can check that p_* is the colimit of the morphism p_{Z*} as Z runs over $\mathcal{Z}$. In fact, taking any cycle α in $c_0(Y_n/S)$, its support T is finite over S ; as $p_n : Y_n \to X$ is separated, $Z = p_n(T)$ is a closed subscheme of X which is finite over S. Obviously, α belongs to $c_0(Z \times_X Y_n/S)$.

Because $\mathcal{Z}$ is a filtering ordered set, it is sufficient to consider the case where p is p_Z and X is Z. Because $c_0(Z/S)$ is additive with respect to Z, we can assume in addition that Z is connected, which finishes the reduction of the second step.

Final step. Now, S is strictly local and X is finite and connected over S. In particular, X is a strictly local scheme. Let x and s be the closed points of X and S, respectively. Under these assumptions, we have the following lemma (whose proof is given below).

Lemma 2.1.7. *For any S-scheme U and any étale S-morphism $f : U \to X$, the canonical morphism:*

$$\begin{array}{rcl} \varphi_U : \mathbf{Z}\langle \mathrm{Hom}_X(X,U)\rangle \otimes c_0(X/S) & \longrightarrow & c_0(U/S) \\ (i : X \to U) \otimes \beta & \longmapsto & i_*(\beta) \end{array}$$

is an isomorphism.

Thus, according to the lemma above, the map (2.1.6.a) is isomorphic to:

$$p_* : \mathbf{Z}\langle \mathrm{Hom}_X(X,Y_\bullet)\rangle \otimes c_0(X/S) \to \mathbf{Z}\langle \mathrm{Hom}_X(X,X)\rangle \otimes c_0(X/S).$$

As p is an étale hypercovering and X is a strictly local scheme, the simplicial set $\mathrm{Hom}_X(X,Y_\bullet)$ is contractible. This readily implies that p_* is a chain homotopy equivalence, which achieves the proof of the proposition. □

Proof of Lemma 2.1.7. We construct an inverse ψ_U to φ_U. Because $c_0(-/S)$ is additive, the (free) abelian group $c_0(U/S)$ is generated by cycles α whose support is connected. Thus it is enough to define ψ_U on cycles $\alpha \in c_0(U/S)$ whose support T is connected.

By definition, T is finite over S. As f is separated, $f(T)$ is closed in X and the induced map $T \to f(T)$ is finite. In particular, the closed point x of X belongs to $f(T)$: we fix a point $t \in T$ such that $f(t) = x$. Then the residual extension $\kappa(t)/\kappa(x)$ is finite. This implies $\kappa(t) \simeq \kappa(x)$ as $\kappa(x)$ is algebraically closed (according to convention at the beginning of the proof). In particular, t is a $\kappa(x)$-section of the special fiber U_x of U at x. As U/Z is étale, this section can be extended uniquely to a section $i : X \to U$ of U/X. Then $i(X)$ is a connected component, meeting T at least at t. This implies $T \subset i(X)$ as T is connected. Thus $\alpha \in c_0(U/S)$ corresponds to an element α_i in $c_0(i(X)/S) \simeq c_0(X/S)$. We put $\psi_U(\alpha) = i \otimes \alpha_i$. The map ψ_U is obviously an inverse to φ_U, and this concludes the proof of the lemma. □

Remark 2.1.8. This proposition fills out a gap in the theory of motivic complexes of Voevodsky which was left open in [VSF00, chap. 5, sec. 3.3]: Voevodsky restricted to the case of a field of finite cohomological dimension.

Note also the following corollary of lemma 2.1.7:

Corollary 2.1.9. *Let X be a scheme and V an étale X-scheme. Let $R_X(V)$ be the étale R-sheaf on Sm_X represented by V. Then the map*

$$R_X(V) \to R_X^{tr}(V)$$

induced by the graph functor is an isomorphism.

Proof. As in the proof above, it is sufficient to treat the case $R = \mathbf{Z}$. Moreover, by looking at the toposic fibers of the above map, and by using the arguments of the first step of the proof, we are reduced to check that the map

$$\mathbf{Z}\langle \mathrm{Hom}_X(X,V)\rangle \to c_0(V/X)$$

is an isomorphism when X is strictly local with algebraically closed residue field. Then, this follows from the preceding lemma, and from the fact that, when X is connected, we have $c_0(X/X) = \mathbf{Z}$; see [CD12, Lemma 10.2.6]. □

In [CD12, Proposition 10.3.3], we proved the preceding proposition in the particular case of a Čech hypercovering – *i.e.* the coskeleton of an étale cover. With the extension obtained in the above proposition, we can apply [CD12, Prop. 9.3.9] and get the following.

Proposition 2.1.10. *The category of étale sheaves with transfers has the following properties.*

(1) *The forgetful functor*

$$\mathscr{O}_{\text{ét}}^{tr} : \mathrm{Sh}_{\text{ét}}^{tr}(S,R) \to \mathrm{PSh}^{tr}(S,R)$$

admits an exact left adjoint $a_{\text{ét}}^{tr}$ such that the following diagram commutes, where $a_{\text{ét}}$ denotes the usual sheafification functor.

$$\begin{tikzcd} \mathrm{PSh}^{tr}(S,R) \arrow[r, "a_{\text{ét}}^{tr}"] \arrow[d, "\hat{\gamma}_*"'] & \mathrm{Sh}_{\text{ét}}^{tr}(S,R) \arrow[d, "\gamma_*"] \\ \mathrm{PSh}(S,R) \arrow[r, "a_{\text{ét}}"] & \mathrm{Sh}_{\text{ét}}(S,R) \end{tikzcd}$$

(2) *The category $\mathrm{Sh}_{\text{ét}}^{tr}(S,R)$ is a Grothendieck abelian category generated by the sheaves of shape $R_S^{tr}(X)$, for any smooth S-scheme X.*

(3) *The functor γ_* is conservative and commutes with every small limits and colimits.*

2.1.11. We deduce immediately from that proposition that the functor γ_* admits a left adjoint γ^*.

As in [CD12, Corollary 10.3.11], we get the following corollary of the above proposition – see Section A.1 for explanation on premotivic categories which where defined in [CD09]:

Corollary 2.1.12. *The category $\mathrm{Sh}_{\text{ét}}^{tr}(-,R)$ has a canonical structure of an abelian premotivic category. Moreover, the adjunction:*

(2.1.12.a) $$\gamma^* : \mathrm{Sh}_{\text{ét}}(-,R) \leftrightarrows \mathrm{Sh}_{\text{ét}}^{tr}(-,R) : \gamma_*$$

is an adjunction of abelian premotivic categories.

2.1.13. Remember that the category of (Nisnevich) sheaves with transfers $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$ is defined as the category of presheaves with transfers F over S such that $F \circ \gamma$ is a sheaf; see [CD12, 10.4.1]. Then $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(-,R)$ is a fibred category which is an abelian premotivic category according to *loc. cit*.

We will denote by τ the comparison functor between the Nisnevich and the étale topology on the site Sm_S. Thus, we denote by $\tau_* : \mathrm{Sh}^{tr}_{\text{ét}}(S,R) \to \mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$ the obvious fully faithful functor. Then the functor $a^{tr}_{\text{ét}} : \mathrm{PSh}^{tr}(S,R) \to \mathrm{Sh}^{tr}_{\text{ét}}(S,R)$ obviously induces a right adjoint τ^* to the functor τ_*. Moreover, this defines an adjunction of premotivic abelian categories:

$$\tau^* : \mathrm{Sh}^{tr}_{\mathrm{Nis}}(-,R) \leftrightarrows \mathrm{Sh}^{tr}_{\text{ét}}(-,R) : \tau_*. \tag{2.1.13.a}$$

2.2. **Derived categories.**

2.2.1. In [CD12, Section 5], we established a theory to study derived categories such as $\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$. This category has to satisfy the technical conditions of [CD12, Definitions 5.1.3 and 5.1.9]. Let us make explicit this definition in our particular case.

Definition 2.2.2. Let K be a complex of étale R-sheaves with transfers.

(1) The complex K is said to be *local* with respect to the étale topology if, for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))}(R^{tr}_S(X)[n],K) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))}(R^{tr}_S(X)[n],K)$$

is an isomorphism.

(2) The complex K is said to be *étale-flasque* if for any étale hypercover $Y_\bullet \to X$ in Sm_S and any integer $n \in \mathbf{Z}$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))}(R^{tr}_S(X)[n],K) \to \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))}(R^{tr}_S(Y_\bullet)[n],K)$$

is an isomorphism.

Proposition 2.2.3. *A complex of étale sheaves with transfers is étale-flasque if and only if it is local with respect to the étale topology. Moreover, for any complex of étale R-sheaves with transfers K over S, any smooth S-scheme X, and any integer $n \in \mathbf{Z}$, we have a natural identification:*

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(X,R))}(R^{tr}_S(X),K[n]) = H^n_{\text{ét}}(X,K).$$

Proof. Note that the analogous statement is known to be true for complexes of étale sheaves without transfers (see for instance [CD09]). Therefore, the first assertion of the proposition follows from the second one, which we will now prove. Let S be a base scheme.

We consider the projective model category structure on the category $\mathrm{C}(\mathrm{Sh}_{\text{ét}}(S,R))$, that is the analog of the model structure defined in 1.1.8: the weak equivalences are the quasi-isomorphisms, while the fibrations are the morphisms of complexes whose restriction to each of the small sites $X_{\text{ét}}$ is a fibration in the sense of 1.1.8 for any smooth S-scheme X. On the other hand, as the category $\mathrm{Sh}^{tr}_{\text{ét}}(S,R)$ is an abelian Grothendieck category, the category $\mathrm{C}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$ is endowed with the injective model category structure; see [CD09, 2.1]. By virtue of [CD09, 2.14], Proposition 2.1.6 and the last assertion of Proposition 2.1.10 imply that the functor

$$\gamma^* : \mathrm{C}(\mathrm{Sh}_{\text{ét}}(S,R)) \to \mathrm{C}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$$

is a left Quillen functor. As its right adjoint γ_* preserves weak equivalences, we thus get an adjunction

$$\mathbf{L}\gamma^* : \mathrm{D}(\mathrm{Sh}_{\text{ét}}(S,R)) \leftrightarrows \mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R)) : \gamma_* .$$

Note that, for any smooth S-scheme X, we have a natural isomorphism

$$\mathbf{L}\gamma^* R_S(X) \simeq R^{tr}_S(X)$$

because $R_S(X)$ is cofibrant. Therefore, for any smooth S-scheme X and for any complex of étale sheaves with transfers K, we have the following identifications (compare with [VSF00, chap. 5, 3.1.9]):

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(X,R))}(R^{tr}_S(X),K[n]) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(X,R))}(\mathbf{L}\gamma^*(R_S(X)),K[n])\\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\text{ét}}(X,R))}(R_S(X),\gamma_*(K)[n])\\ &= H^n_{\text{ét}}(X,K).\end{aligned}$$

This proves the second assertion of the proposition, and thus achieves its proof. □

2.2.4. Propositions 2.1.6 and 2.2.3 assert precisely that the premotivic abelian category $\mathrm{Sh}^{tr}_{\text{ét}}(-,R)$ is *compatible with the étale topology* in the sense of [CD12, Definition 5.1.9].

We can therefore apply the general machinery of *loc. cit.* to the abelian premotivic category $\mathrm{Sh}^{tr}_{\text{ét}}(-,R)$. In particular, we get triangulated premotivic categories (again, see Section A.1 for basic definitions on premotivic categories):

- [CD12, Definition 5.1.17]: The associated derived category: $\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(-,R))$ whose fiber over a scheme S is $\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$.
- [CD12, Definition 5.2.16]: The associated effective $\mathbf{A}^1$-derived category:

  $$\mathrm{DM}^{eff}_{\text{ét}}(-,R) := \mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}^{tr}_{\text{ét}}(-,R))$$

  whose fiber over a scheme S is the $\mathbf{A}^1$-localization of the derived category $\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$.
  We will call it the category of *effective étale motives*.
- [CD12, Definition 5.3.22]: The associated (stable) $\mathbf{A}^1$-derived category:

  $$\mathrm{DM}_{\text{ét}}(-,R) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}^{tr}_{\text{ét}}(-,R))$$

  whose fiber over a scheme S is obtained from $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R))$ by $\otimes$-inverting the Tate object $R^{tr}_S(1) := \tilde{R}^{tr}_S(\mathbf{P}^1_S,\infty)[-2]$ (in the sense of model categories).
  We will call it the category of *étale motives*.

By construction, these categories are related by the following morphisms of premotivic triangulated categories:

$$\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R)) \xrightarrow{\pi_{\mathbf{A}^1}} \mathrm{DM}^{eff}_{\text{ét}}(S,R) \xrightarrow{\Sigma^\infty} \mathrm{DM}_{\text{ét}}(S,R). \tag{2.2.4.a}$$

Recall that the right adjoint to the functor $\pi_{\mathbf{A}^1}$ is fully faithful with essential image made by the $\mathbf{A}^1$-local complexes, in the sense of the next definition.

Definition 2.2.5. Let K be a complex of R-sheaves with transfers over a scheme S. For any smooth S-scheme X and any integer $n \in \mathbf{Z}$, we simply denote by $H^n_{\text{ét}}(X,K)$ the cohomology of K seen as a complex of R-sheaves over $X_{\text{ét}}$.

We say that K is $\mathbf{A}^1$-*local* if for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, the map induced by the canonical projection

$$H^n_{\text{ét}}(X,K) \to H^n_{\text{ét}}(\mathbf{A}^1_X,K)$$

is an isomorphism.

2.2.6. According to [CD12, 5.1.23, 5.2.19, 5.3.28], the adjunction of abelian premotivic categories (2.1.12.a) can be derived, and it induces, over a scheme S, a commutative diagram:

$$\begin{tikzcd} \mathrm{D}(\mathrm{Sh}_{\text{ét}}(S,R)) \arrow[r] \arrow[d, "\mathbf{L}\gamma^*"'] & \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(S,R)) \arrow[r] \arrow[d] & \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(S,R)) \arrow[d] \\ \mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R)) \arrow[r] & \mathrm{DM}^{\mathit{eff}}_{\text{ét}}(S,R) \arrow[r] & \mathrm{DM}_{\text{ét}}(S,R) \end{tikzcd} \tag{2.2.6.a}$$

Note that all the vertical maps are obtained by deriving (on the left) the functor γ^*. We will simply denote these maps by $\mathbf{L}\gamma^*$. By definition, they admit a right adjoint that we denote by $\mathbf{R}\gamma_*$. In fact, we will often write $\mathbf{R}\gamma_* = \gamma_*$ because of the following simple result.

Proposition 2.2.7. *The exact functor* $\gamma_* : \mathrm{C}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R)) \to \mathrm{C}(\mathrm{Sh}_{\text{ét}}(S,R))$ *preserves* $\mathbf{A}^1$-*equivalences.*

Proof. This follows from [CD12, Proposition 5.2.24]. □

2.2.8. Applying again [CD12, 5.1.23, 5.2.19, 5.3.28] to the adjunction (2.1.13.a), we get a commutative diagram of left derived functors:

$$\begin{tikzcd} \mathrm{D}(\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)) \arrow[r] \arrow[d, "\mathbf{L}\tau^*"'] & \mathrm{DM}^{\mathit{eff}}(S,R) \arrow[r] \arrow[d] & \mathrm{DM}(S,R) \arrow[d] \\ \mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,R)) \arrow[r] & \mathrm{DM}^{\mathit{eff}}_{\text{ét}}(S,R) \arrow[r] & \mathrm{DM}_{\text{ét}}(S,R) \end{tikzcd} \tag{2.2.8.a}$$

where $\mathrm{DM}^{\mathit{eff}}(S,R)$ (resp. $\mathrm{DM}(S,R)$) stands for the effective category (resp. stable category) of Nisnevich motives as defined in [CD12, Definition 11.1.1].

The following proposition is a generalization of [VSF00, chap. 5, 4.1.12].

Proposition 2.2.9. *Assume R is a* $\mathbf{Q}$-*algebra. Then the adjunction* (2.1.13.a) *is an equivalence of categories. In particular, all the vertical maps of the diagram* (2.2.8.a) *are equivalences of categories.*

Proof. We first prove that the right adjoint τ_* of (2.1.13.a) is exact. Using the analog of Proposition 2.2.3 for the Nisnevich topology, one reduces to show that for any étale R-sheaf with transfers F over S and any local henselian scheme X over S, the cohomology group $H^1_{\text{ét}}(X,F)$ vanishes. But, as F is rational, this last group is isomorphic to $H^1_{\mathrm{Nis}}(X,F)$ – this is well known, see for example [CD12, 10.5.9] – and this group is zero.

Note also τ_* obviously commutes with direct sums. Thus it commutes with arbitrary colimits.

Obviously τ_* is essentially surjective. It remains only to prove it is fully faithful. Thus, we have to prove that for any Nisnevich R-sheaf with transfers over S, the adjunction map

$$F_{\text{ét}} = \tau^*\tau_*(F) \to F$$

is an isomorphism. As $\tau^*\tau_*$ commutes with colimits, it is sufficient to prove this for $F = R^{tr}_S(X)$ when X is an arbitrary smooth S-scheme. This is precisely Proposition 2.1.4. □

2.3. **A weak localization property.**

Lemma 2.3.1. *Let $f : Y \to X$ be a finite morphism. Then the functor*

$$f_* : \mathrm{C}(\mathrm{Sh}^{tr}_{\text{ét}}(Y,R)) \to \mathrm{C}(\mathrm{Sh}^{tr}_{\text{ét}}(X,R))$$

preserves colimits and $\mathbf{A}^1$-equivalences.

Proof. We first check that f_* preserves colimits. By definition, $\gamma_* f_* = f_* \gamma_*$. According to point (3) of Proposition 2.1.10, we thus are reduced to prove the functor $f_* : \mathrm{Sh}(Y,R) \to \mathrm{Sh}(X,R)$ commutes with colimits. This is well known – boiling down to the fact a finite scheme over a strictly local scheme is a sum of strictly local schemes. The remaining assertion now follows from [CD12, Prop. 5.2.24]. □

Proposition 2.3.2. *Let $f : Y \to X$ be a finite morphism. Then the functor*

$$f_* = \mathbf{R}f_* : \mathrm{DM}^{eff}_{\text{ét}}(Y,R) \to \mathrm{DM}^{eff}_{\text{ét}}(X,R)$$

preserves small sums, and thus, has a right adjoint $f^!$.

Proof. The fact that the functor f_* preserves small sums follows formally from the preceding lemma and from the fact that $\mathbf{A}^1$-equivalences are closed under filtered colimits; see [CD09, Proposition 4.6]. The existence of the right adjoint $f^!$ follows from the Brown representability theorem[2]. □

2.3.3. Let $i : Z \to S$ be a closed immersion and $j : U \to S$ the complementary open immersion.

Let K be a complex of étale sheaves with transfers over S. Note that the composite of the obvious adjunction maps

$$j_\sharp j^*(K) \to K \to i_* i^*(K) \tag{2.3.3.a}$$

is always 0. We will say that this sequence is *homotopy exact in* $\mathrm{DM}^{eff}_{\text{ét}}(S,R)$ if for any cofibrant resolution $K' \to K$ of K the canonical map

$$\mathrm{Cone}\big(j_\sharp j^*(K') \to K'\big) \to i_* i^*(K')$$

is an $\mathbf{A}^1$-equivalence.

Note that given a smooth S-scheme X, $K = R^{tr}_S(X)$ is cofibrant by definition and the cone appearing above is quasi-isomorphic to the cokernel of the map

$$R^{tr}_S(X - X_Z) \xrightarrow{j_*} R^{tr}_S(X),$$

which we will denote by $R^{tr}_S(X/X - X_Z)$. Here, we put $X_Z = X \times_S Z$.

We recall the following proposition from [CD12, Cor. 2.3.17]:

Proposition 2.3.4. *Consider the notations above. The following conditions are equivalent:*

(i) *The functor i_* is fully faithful and the pair of functors (i^*, j^*) is conservative for the premotivic category $\mathrm{DM}^{eff}_{\text{ét}}(-,R)$.*

(ii) *For any complex K, the sequence* (2.3.3.a) *is homotopy exact in* $\mathrm{DM}^{eff}_{\text{ét}}(S,R)$.

[2] One can see the existence of a right adjoint of $\mathbf{R}f_*$ in a slightly more constructive way as follows. Lemma 2.3.1 implies that the functor $f^!$ already exists at the level of étale sheaves with transfers. One can see easily from the same lemma that f_* is a left Quillen functor with respect to the $\mathbf{A}^1$-localizations of the injective model category structures, which ensures the existence of $f^!$ at the level of the homotopy categories, namely as the total right derived functor of its analog at the level of sheaves.

(iii) *The functor i_* commutes with twists and for any smooth S-scheme X, the canonical map*

$$R_S^{tr}(X/X-X_Z) \to i_*(R_Z^{tr}(X_Z))$$

is an isomorphism in $\mathrm{DM}^{eff}_{\text{ét}}(S,R)$.

Moreover, when these conditions are fulfilled, for any complex K, the exchange transformation:

$$(i_*(R_Z)) \otimes K \to i_* i^*(K) \tag{2.3.4.a}$$

is an isomorphism.

The equivalent conditions of the above proposition are called the *localization property with respect to i* for the premotivic triangulated category $\mathrm{DM}^{eff}_{\text{ét}}(-,R)$; see A.1.11.

Proposition 2.3.5. *Let $i : Z \to S$ be a closed immersion which admits a smooth retraction $p : S \to Z$. Then $\mathrm{DM}^{eff}_{\text{ét}}(-,R)$ satisfies the localization property with respect to i.*

The proof of this proposition is the same than the analogous fact for the Nisnevich topology – see [CD12, Prop. 6.3.14]. As this statement plays an important role in the sequel of these notes, we will recall the essential steps of the proof. One of the main ingredients of the proof uses the following result, proved in [Ayo07, 4.5.44]:

Theorem 2.3.6. *The premotivic category $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(-,R))$ satisfies localization (with respect to any closed immersion).*

Lemma 2.3.7. *For any open immersion $j : U \to S$, the exchange transformation*

$$\mathbf{L}j_\sharp \gamma_* \to \gamma_* \mathbf{L}j_\sharp$$

is an isomorphism in $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(S,R))$.

Proof. We first prove that, for any étale sheaf with transfers F over U, the map

$$j_\sharp \gamma_*(F) \to \gamma_* j_\sharp(F)$$

is an isomorphism of étale sheaves. Indeed, both in the case of étale sheaves or of étale sheaves with transfers, the sheaf $j_\sharp(F)$ is obtained as the sheaf associated with the presheaf

$$V \mapsto \begin{cases} F(V) & \text{if } V \text{ is supported over } U \text{ (i.e. if } V \times_S U \simeq V\text{)}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the functors $j_\sharp$ are exact, and they preserve $\mathbf{A}^1$-equivalences because of the projection formula $A \otimes j_\sharp(B) \simeq j_\sharp(j^*(A) \otimes B)$ (for any sheaves A and B). Using Proposition 2.2.7, this implies the lemma. □

Lemma 2.3.8. *let $i : Z \to S$ be a closed immersion which admits a smooth retraction. Then the exchange transformation:*

$$\mathbf{L}\gamma^* i_* \to i_* \mathbf{L}\gamma^*$$

is an isomorphism in $\mathrm{DM}^{eff}_{\text{ét}}(S,R)$.

Proof. Let $p : S \to Z$ be a smooth morphism such that $pi = 1_Z$, and denote by $j : U \to S$ the complement of i in S. For any object M in $\mathrm{DM}^{eff}_{\text{ét}}(Z,R)$, we have a natural homotopy cofiber sequence of shape

(2.3.8.a) $$\mathbf{L}j_\sharp j^* p^* M \to p^* M \to i_* M$$

(note that $i_* M = i_* i^* p^* M$ because $pi = 1_Z$). Indeed, as the functor γ_* is conservative, it is sufficient to check this after applying γ_*. As the functor γ_* commutes with $\mathbf{L}j_\sharp$ (by the previous lemma) as well as with the functors j^*, p^* and i_* (because its left adjoint $\mathbf{L}\gamma^*$ commutes with the functors $\mathbf{L}j_\sharp$, $\mathbf{L}p_\sharp$ and $\mathbf{L}i^*$), it is sufficient to see that the analogue of (2.3.8.a) is an homotopy cofiber sequence for any object M of $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Z,R))$. But this latter property is a particular case of the localization property with respect to the closed immersions, which is known to hold by Theorem 2.3.6. The characterization of the functor i_* by the homotopy cofiber sequence (2.3.8.a) implies the lemma because the functor $\mathbf{L}\gamma^*$ is known to commute with the functors $\mathbf{L}j_\sharp$, j^* and p^*. □

Proof of Proposition 2.3.5. Now, the proposition can easily be deduced from the above lemma and from Theorem 2.3.6, using the fact that the functor γ_* is conservative; see the proof of [CD12, Prop. 6.3.14] for more details. □

3. THE EMBEDDING THEOREM

3.1. **Locally constant sheaves and transfers.**

3.1.1. Let X be a scheme.

Recall that we denote by $\mathrm{Sh}(X_{\text{ét}},R)$ the category of R-sheaves over the small étale site $X_{\text{ét}}$. On the other hand, we also have the category $\mathrm{Sh}_{\text{ét}}(X,R)$ of R-sheaves over the smooth-étale site $Sm_{X,\text{ét}}$ – made by smooth X-schemes. The obvious inclusion of sites $\rho : X_{\text{ét}} \to Sm_{X,\text{ét}}$ gives an adjunction of categories:

(3.1.1.a) $$\rho_\sharp : \mathrm{Sh}(X_{\text{ét}},R) \rightleftarrows \mathrm{Sh}_{\text{ét}}(X,R) : \rho^*$$

where $\rho^*(F) = F \circ \rho$. The following lemma is well known (see [AGV73, VII, 4.0, 4.1]):

Lemma 3.1.2. *With the above notations, the following properties hold:*

(1) *the functor ρ^* commutes with arbitrary limits and colimits;*
(2) *the functor $\rho_\sharp$ is exact and fully faithful;*
(3) *the functor $\rho_\sharp$ is monoidal and commutes with operations f^* for any morphism of schemes f, and with $f_\sharp$, when f is étale.*

Note that point (3) can be rephrased by saying that (3.1.1.a) is an adjunction of étale-premotivic abelian categories (Definition A.1.7).

By definition, $\rho_\sharp$ sends the R-sheaf on $X_{\text{ét}}$ represented by an étale X-scheme V to the R-sheaf represented by V on Sm_X. We will denote by $R_X(V)$ both the sheaves on the small étale and on the smooth-étale site of X – the confusion here is harmless.

3.1.3. Let us denote by $\mathrm{D}(X_{\text{ét}},R)$ the derived category of $\mathrm{Sh}(X_{\text{ét}},R)$. As both functors $\rho_\sharp, \rho^*$ are exact, they can be derived trivially. In particular, we get a derived adjunction:

(3.1.3.a) $$\rho_\sharp : \mathrm{D}(X_{\text{ét}},R) \rightleftarrows \mathrm{D}(\mathrm{Sh}_{\text{ét}}(X,R)) : \rho^*$$

in which the functor $\rho_\sharp$ is still fully faithful.

Proposition 3.1.4. *The composite functor*

$$\mathrm{Sh}(X_{\acute{e}t},R) \xrightarrow{\rho_\sharp} \mathrm{Sh}_{\acute{e}t}(X,R) \xrightarrow{\gamma^*} \mathrm{Sh}^{tr}_{\acute{e}t}(X,R)$$

is exact and fully faithful.

Proof. As $\rho_\sharp$ is fully faithful and γ_* is exact and conservative, it is sufficient to prove that, for any R-sheaf F on $X_{\acute{e}t}$, the map induced by adjunction:

$$\rho_\sharp(F) \to \gamma_*\gamma^*\rho_\sharp(F)$$

is an isomorphism of étale sheaves. Moreover, all the involved functors commute with colimits (applying in particular 2.1.10). Thus, it is sufficient to prove this in the case where $F = R_X(V)$ is representable by an étale X-scheme V. Then, the result is just a reformulation of Corollary 2.1.9. □

Corollary 3.1.5. *The functor*

$$\mathbf{L}\gamma^*\rho_\sharp = \gamma^*\rho_\sharp : \mathrm{D}(X_{\acute{e}t},R) \to \mathrm{D}(\mathrm{Sh}^{tr}_{\acute{e}t}(X,R))$$

is fully faithful.

3.1.6. We have a composite functor

$$\rho_! : \mathrm{D}(X_{\acute{e}t},R) \to \mathrm{D}(\mathrm{Sh}^{tr}_{\acute{e}t}(X,R)) \to \mathrm{DM}^{eff}_{\acute{e}t}(X,R) \tag{3.1.6.a}$$

Proposition 3.1.7. *Assume that the ring R is of positive characteristic n and that the residue characteristics of X are prime to n. Then the composed functor* (3.1.6.a) *is fully faithful.*

Proof. Recall that the functor $\pi_{\mathbf{A}^1} : \mathrm{D}(\mathrm{Sh}^{tr}_{\acute{e}t}(X,R) \to \mathrm{DM}^{eff}_{\acute{e}t}(X,R)$ has a fully faithful right adjoint whose essential image consists of $\mathbf{A}^1$-local objects (see Definition 2.2.5). Therefore, by virtue of Proposition 2.2.3 and of Corollary 3.1.5, it is sufficient to prove that, for any complex K in $\mathrm{D}(X_{\acute{e}t},R)$, and for any étale X-scheme V, the map

$$H^i_{\acute{e}t}(V,K) \to H^i_{\acute{e}t}(\mathbf{A}^1 \times V,K)$$

is bijective for all i, which is Theorem 1.3.2. □

3.2. **Etale motivic Tate twist.** Recall from [AGV73, IX, 3.2] that, for any scheme X such that n is invertible in $\mathcal{O}_X$, the group scheme $\mu_{n,X}$ of nth roots of unity fits in the Kummer short exact sequence in $\mathrm{Sh}_{\acute{e}t}(S,\mathbf{Z})$:

$$0 \to \mu_n \to \mathbf{G}_{m,X} \to \mathbf{G}_{m,X} \to 0. \tag{3.2.0.a}$$

This induces a canonical isomorphism in the derived category:

$$\mathbf{G}_{m,X}[-1] \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z} \simeq \mu_{n,X}. \tag{3.2.0.b}$$

3.2.1. For any scheme S and any ring R, the Tate motive $R_S(1)$ is defined in $\mathrm{DM}^{eff}_{\acute{e}t}(S,R)$ as the cokernel of the split monomorphism $R^{tr}_S(S)[-1] \to R^{tr}_S(\mathbf{G}_{m,S})[-1]$ induced by the unit section.

As $\mathbf{G}_{m,S}$ has a natural structure of étale sheaf with transfers, there is a canonical map

$$\mathbf{Z}^{tr}_S(\mathbf{G}_{m,S}) \to \mathbf{G}_{m,S}$$

which factor through $\mathbf{Z}_S(1)[1]$. This gives a natural morphism in $\mathrm{DM}^{eff}(S,R)$:

$$R_S(1)[1] \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R\,. \tag{3.2.1.a}$$

In the case where R is of positive characteristic n, with n invertible in $\mathcal{O}_S$, the isomorphism (3.2.0.b) identifies the map (3.2.1.a) shifted by $[-1]$ with a morphism of shape

$$R_S(1) \to \mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R\,, \tag{3.2.1.b}$$

where the locally constant étale sheaf $\mu_{n,S}$ is considered as a sheaf with transfers (according to proposition 3.1.7). Note also that $\mu_{n,S} \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R \simeq \mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R$ because μ_n is a locally free sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules.

Proposition 3.2.2. *The morphism* (3.2.1.a) *is an isomorphism in* $\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(S,R)$ *whenever S is regular.*

Proof. The case where $R = \mathbf{Z}$ follows immediately from [CD12, Theorem **??**]. We conclude in general by applying the derived functor $(-) \otimes^{\mathbf{L}} R$. □

Proposition 3.2.3. *If the ring R is of positive characteristic n, with n invertible in $\mathcal{O}_S$, then the morphism* (3.2.1.b) *is an isomorphism in* $\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(S,R)$.

Proof. By virtue of the preceding proposition, this is true for S regular, and thus in the case where $S = \operatorname{Spec} \mathbf{Z}[1/n]$. Now, consider a morphism of schemes $f : X \to S$, with S regular (e.g. $S = \operatorname{Spec} \mathbf{Z}[1/n]$). The natural map $\mathbf{L}f^*(R_S(1)) \to R_X(1)$ is obviously an isomorphism, and, as the étale sheaf μ_n is locally constant, the canonical map $\mathbf{L}f^*(\mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R) \to \mu_{n,X} \otimes_{\mathbf{Z}/n\mathbf{Z}} R$ is invertible as well, from which we deduce the general case. □

Corollary 3.2.4. *For any scheme X, if n is invertible in $\mathcal{O}_X$, we have a canonical identification:*

$$\operatorname{Hom}_{\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,\mathbf{Z}/n\mathbf{Z})}((\mathbf{Z}/n\mathbf{Z})_X, (\mathbf{Z}/n\mathbf{Z})_X(1)[i]) = H^{i-1}_{\text{ét}}(X, \mu_n)\,.$$

Proof. This is an immediate consequence of Propositions 3.1.7 and 3.2.3. □

Corollary 3.2.5. *If the ring R is of positive characteristic n, with n prime to the residue characteristics of X, then the Tate twist $R_X(1)$ is $\otimes$-invertible in* $\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)$. *Therefore, the infinite suspension functor* (2.2.4.a)

$$\Sigma^\infty : \mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R) \to \mathrm{DM}_{\text{ét}}(X,R)$$

is then an equivalence of categories.

Proof. The sheaf $\mu_{n,X}$ is locally constant: there exists an étale cover $f : Y \to X$ such that $f^*(\mu_{n,X}) = (\mathbf{Z}/n\mathbf{Z})_Y$. This implies that the sheaf $\mu_{n,X} \otimes R$ is $\otimes$-invertible in the derived category $\mathrm{D}(X_{\text{ét}},R)$. As the canonical functor $\mathrm{D}(X_{\text{ét}},R) \to \mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)$ is symmetric monoidal, this implies that $\mu_{n,X} \otimes R$ is $\otimes$-invertible in $\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)$. The first assertion follows then from Proposition 3.2.3. The second follows from the first by the general properties of the stabilization of model categories; see [Hov01]. □

4. Torsion étale motives

4.0.6. In this section, we fix a ring R of positive characteristic n. Our category of underlying schemes *Sch* will be the category of all noetherian schemes. We will denote by $Sch[1/n]$ the category of $\mathbf{Z}[1/n]$-schemes.

The aim of this section is to show that the premotivic triangulated category of R-linear étale motives $\mathrm{DM}^{eff}_{\text{ét}}(-,R)$ defined previously satisfies the Grothendieck 6 functors formalism as well as the absolute purity property (see respectively Definitions A.1.10 and A.2.9). Then we deduce the extension of the Suslin-Voevodsky rigidity theorem [VSF00, chap. 5, 3.3.3] to arbitrary bases.

To simplify notations, we will cancel the letters $\mathbf{L}$ and $\mathbf{R}$ in front of the derived functors used in this section. Note also that we will show in Proposition 4.1.1 that

$$\Sigma^\infty : \mathrm{DM}^{eff}_{\text{ét}}(-,R) \to \mathrm{DM}_{\text{ét}}(-,R)$$

is an equivalence of categories. Thus we will use the simpler notation $\mathrm{DM}_{\text{ét}}(-,R)$ from section 4.2 on.

4.1. **Stability and orientation.** We first show that in Corollary 3.2.5 one can drop the restriction on the characteristic of the schemes we consider:

Proposition 4.1.1. *If R is of positive characteristic, for any scheme S the Tate motive $R_S(1)$ in $\otimes$-invertible and the natural map $R_S(1)[1] \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$ (3.2.1.a) is an isomorphism in $\mathrm{DM}^{eff}_{\text{ét}}(S,R)$.*

Proof. Let $n > 0$ be the characteristic of R. As the change of scalars functor

$$\mathrm{DM}^{eff}_{\text{ét}}(S,\mathbf{Z}/n\mathbf{Z}) \to \mathrm{DM}^{eff}_{\text{ét}}(S,R)\,, \quad M \mapsto R \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} M$$

is symmetric monoidal, it is sufficient to prove this for $R = \mathbf{Z}/n\mathbf{Z}$. By a simple devissage, we may assume that $n = p^\alpha$ is some power of a prime number p. Let $S[1/p]$ be the product $S \times \mathrm{Spec}(\mathbf{Z}[1/p])$, and let $j : S[1/p] \to S$ be the canonical open immersion. By virtue of Proposition A.3.4, the functor

$$j^* : \mathrm{DM}^{eff}_{\text{ét}}(S,R) \to \mathrm{DM}^{eff}_{\text{ét}}(S[1/p],R)$$

is an equivalence of triangulated monoidal categories. Therefore, we may also assume that n is invertible in $\mathscr{O}_S$. We are thus reduced to Corollary 3.2.5. □

Corollary 4.1.2. *If R is a ring of positive characteristic, then, for any scheme S the infinite suspension functor*

$$\Sigma^\infty : \mathrm{DM}^{eff}_{\text{ét}}(S,R) \to \mathrm{DM}_{\text{ét}}(S,R)$$

is an equivalence of categories.

4.1.3. If R is of positive characteristic, as a direct consequence of the Proposition 4.1.1, we have, for any scheme S, a functorial morphism of abelian groups

$$c_1^{\text{ét}} : \mathrm{Pic}(S) = \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}_{\text{ét}}(S,\mathbf{Z}))}(\mathbf{Z}_S, \mathbf{G}_{m,S}[1]) \to \mathrm{Hom}_{\mathrm{DM}^{eff}(S,R)}(R_S, R_S(1)[2])$$

which is simply induced by the canonical morphism $\mathbf{G}_{m,S} \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$ and the isomorphism $R_S(1)[1] \simeq \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$.

Definition 4.1.4. We call the map $c_1^{\text{ét}}$ the *étale motivic Chern class*.

We will consider this map as the canonical orientation of the triangulated premotivic category $\mathrm{DM}^{eff}_{\text{ét}}(-,R)$.

4.2. **Purity (smooth projective case).**

4.2.1. We need to simplify some of our notations which will often appear below. Given any morphism f and any smooth morphism p, we will consider the following unit and counit maps of the relevant adjunctions in $\mathrm{DM}_{\text{ét}}(-,R)$:

$$\begin{aligned} 1 &\xrightarrow{\alpha_f} f_* f^*, \qquad & f^* f_* &\xrightarrow{\alpha'_f} 1, \\ 1 &\xrightarrow{\beta_p} p^* p_\sharp, \qquad & p_\sharp p^* &\xrightarrow{\beta'_p} 1. \end{aligned} \tag{4.2.1.a}$$

Remark 4.2.2. Consider a cartesian square of schemes:

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"] & S \end{tikzcd}$$

such that p is smooth. According to Property (5) of Definition A.1.1, applied to $\mathrm{DM}_{\text{ét}}(-,R)$, we associate to the square Δ the base change isomorphism

$$Ex(\Delta_\sharp^*) : q_\sharp g^* \to f^* p_\sharp.$$

In what follows, the square Δ will be clear and we will put simply: $Ex_\sharp^* := Ex(\Delta_\sharp^*)^{-1}$.

Recall also that we associate to the square Δ another *exchange transformation* as the following composite (see [CD12, 1.1.15]):

$$Ex_{\sharp *} : p_\sharp g_* \xrightarrow{\alpha_f} f_* f^* p_\sharp g_* \xrightarrow{Ex_\sharp^*} f_* q_\sharp g^* g_* \xrightarrow{\alpha'_g} f_* q_\sharp. \tag{4.2.2.a}$$

4.2.3. Proposition 4.1.1, and the existence of the map $c_1^{\text{ét}}$ defined in 4.1.4, show that the category $\mathrm{DM}_{\text{ét}}(S,R)$ satisfies all the assumptions of [Dég07, §2.1]. Thus, the results of this article can be applied to that latter category. In particular, according to Prop. 4.3 of *op. cit.*, we get:

Proposition 4.2.4. *Assume that the ring R is of positive characteristic. Let $f : X \to S$ be a smooth morphism of pure dimension d and $s : S \to X$ be a section of f. Then, using the notation of 2.3.3, there exists a canonical isomorphism in* $\mathrm{DM}_{\text{ét}}(S,R)$*:*

$$\mathfrak{p}'_{f,s} : R_S^{tr}(X/X-S) \to R_S(d)[2d].$$

In particular, for any motive K in $\mathrm{DM}_{\text{ét}}(S,R)$, we get a canonical isomorphism:

$$\mathfrak{p}_{f,s} : \begin{cases} f_\sharp s_*(K) = f_\sharp s_*(s^* f^*(K) \otimes R_S) & \xrightarrow{\sim} K \otimes f_\sharp s_*(R_S) \\ & = K \otimes R_S^{tr}(X/X-S) \xrightarrow{\mathfrak{p}'_{f,s}} K(d)[2d] \end{cases}$$

which is natural in K. The first isomorphism uses the projection formulas respectively for the smooth morphism f (see point (5) of Definition A.1.1) and for the immersion s (*i.e.* the isomorphism (2.3.4.a)).

4.2.5. Assume now that $f : X \to S$ is smooth and projective of dimension d. We consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\delta"] & X \times_S X \arrow[r, "f''"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Theta"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

where Θ is the obvious cartesian square and δ is the diagonal embedding.

As in [CD12, 2.4.39], we introduce the following natural transformation:

$$\mathfrak{p}_f : f_\sharp = f_\sharp f''_* \delta_* \xrightarrow{Ex_{\sharp *}} f_* f'_\sharp \delta_* \xrightarrow{\mathfrak{p}_{f',\delta}} f_*(d)[2d] \tag{4.2.5.a}$$

with the notation of Remark 4.2.2 with respect to the square Θ.

Theorem 4.2.6. *Under the above assumptions, the map $\mathfrak{p}_f$ is an isomorphism.*

Proof. In this proof, we put $\tau(K) = K(d)[2d]$. Note that according to the basic properties of a premotivic category, we get the following identification of funtors for $\mathrm{DM}_{\text{ét}}(-,R)$:

$$f^*\tau = \tau f^*, f_\sharp \tau = \tau f_\sharp. \tag{4.2.6.a}$$

Moreover, we can define a natural *exchange transformation*:

$$Ex_\tau : \tau f_* \xrightarrow{\alpha_f} f_* f^* \tau f_* = f_* \tau f^* f_* \xrightarrow{\alpha'_f} f_* \tau \tag{4.2.6.b}$$

with the notations of Paragraph 4.2.1. Using the fact τ is an equivalence of categories according to Proposition 4.1.1, we deduce easily from the identification (4.2.6.a) that τ_f is an isomorphism.

The key point of the proof is the following lemma inspired by a proof of J. Ayoub (see the proof of [Ayo07, 1.7.14, 1.7.15]):

Lemma 4.2.7. *To check that $\mathfrak{p}_f$ is an isomorphism, it is sufficient to prove that the natural transformation*

$$\mathfrak{p}_f.f^* : f_\sharp f^* \to f_* \tau f^*$$

is an isomorphism.

To prove the lemma we construct a right inverse ϕ_1 and a left inverse ϕ_2 to the morphism $\mathfrak{p}_f$ as the following composite maps:

$$\phi_1 : f_*\tau \xrightarrow{\alpha_f} f_* f^* f_* \tau \xrightarrow{Ex_\tau^{-1}} f_* f^* \tau f_* = f_* \tau f^* f_* \xrightarrow{(\mathfrak{p}_f.f^*f_*)^{-1}} f_\sharp f^* f_* \xrightarrow{\alpha'_f} f_\sharp$$

$$\phi_2 : f_*\tau \xrightarrow{\beta_f} f_* \tau f^* f_\sharp \xrightarrow{(\mathfrak{p}_f.f^*f_\sharp)^{-1}} f_\sharp f^* f_\sharp \xrightarrow{\beta'_f} f_\sharp.$$

Let us check that $\mathfrak{p}_f \circ \phi_1 = 1$. To prove this relation, we prove that the following diagram is commutative:

$$\begin{tikzcd}
f_*\tau \arrow[r, "\alpha_f"] \arrow[ddd, equal] & f_*f^*f_*\tau \arrow[r, "Ex_\tau^{-1}"] \arrow[dd, equal] & f_*\tau f^*f_* \arrow[r, "(\mathfrak{p}_f f^*f_*)^{-1}"] \arrow[d, equal] & f_\sharp f^*f_* \arrow[r, "\alpha'_f"] \arrow[d, equal] & f_\sharp \arrow[r, "\mathfrak{p}_f"] \arrow[d, phantom, "(1)"] & f_*\tau \arrow[d, equal] \\
 & & f_*\tau f^*f_* \arrow[r, "(\mathfrak{p}_f f^*f_*)^{-1}"] \arrow[d, equal] & f_\sharp f^*f_* \arrow[r, "\mathfrak{p}_f f^*f_*"] \arrow[d, phantom, "(2)"] & f_*\tau f f_* \arrow[r, "\alpha'_f"] & f_*\tau \arrow[d, equal] \\
 & f_*f^*f_*\tau \arrow[r, "Ex_\tau^{-1}"] \arrow[d, equal] & f_*\tau f^*f_* \arrow[rrr, "\alpha'_f" description] \arrow[d, phantom, "(3)"] & & & f_*\tau \arrow[d, equal] \\
f_*\tau \arrow[r, "\alpha_f"] & f_*f^*f_*\tau \arrow[rrrr, "\alpha'_f" description] & & & & f_*\tau.
\end{tikzcd}$$

The commutativity of (1) and (2) is obvious and the commutativity of (3) follows from Formula (4.2.6.b) defining Ex_τ. Then the result follows from the usual formula between the unit and counit of an adjunction. The relation $\phi_2 \circ \mathfrak{p}_f = 1$ is proved using the same kind of computations.

The end of the proof now relies on the following lemma which follows from [Dég07, Theorem 5.23], which can be applied, according to Paragraph 4.2.3:

Lemma 4.2.8. *Let $f : X \to S$ be smooth projective of dimension d as above, and $\delta : X \to X \times_S X$ the diagonal embedding. Then the following holds:*

- *The étale motive $R^{tr}_S(X)$ is strongly dualizable in $\mathrm{DM}_{\text{ét}}(S,R)$.*
- *Consider the morphism μ defined by the following composition:*

$$\begin{aligned} R^{tr}_S(X) \otimes_S R^{tr}_S(X) = R^{tr}_S(X \times_S X) \xrightarrow{\pi} R^{tr}_S(X \times_S X / X \times_S X - \delta(X)) \\ \xrightarrow{\mathfrak{p}'_{f',\delta}} R^{tr}_S(X)(d)[2d] \xrightarrow{f_*} R_S(d)[2d]. \end{aligned} \tag{4.2.8.a}$$

where π is the canonical map and $\mathfrak{p}'_{f',\delta}$ is the purity isomorphism of Proposition 4.2.4. Then μ induces by adjunction an isomorphism of endofunctors of $\mathrm{DM}_{\text{ét}}(S,R)$:

$$\big(R^{tr}_S(X) \otimes_S -\big) \xrightarrow{d_{X/S}} \underline{\mathrm{Hom}}(R^{tr}_S(X), -(d)[2d]).$$

To finish the proof, we now check that the map

$$f_\sharp f^* \xrightarrow{\mathfrak{p}_f f^*} f_* \tau f^* = f_* f^* \tau$$

is an isomorphism. Recall that according to the smooth projection formula for the premotivic category $\mathrm{DM}_{\text{ét}}$, we get an identification of functors:

$$f_\sharp f^* = (R^{tr}_S(X) \otimes -).$$

Thus the right adjoint $f_* f^*$ is identified with $\underline{\mathrm{Hom}}(R^{tr}_S(X), -)$. According to the above theorem, it is sufficient to prove that the map $\mathfrak{p}_f f^*$ above coincide through these identifications with the isomorphism $d_{X/S}$ above.

According to the above definition of μ, the natural transformation of functors $(\mu \otimes -)$ can be described as the following composite:

$$\begin{aligned} f_\sharp f^* f_\sharp f^* \xrightarrow{Ex^*_\sharp} f_\sharp f'_\sharp f''^* f^* = g_\sharp g^* \xrightarrow{\alpha_\delta} g_\sharp \delta_* \delta^* g^* \\ = f_\sharp f'_\sharp \delta_* f^* \xrightarrow{\mathfrak{p}_{f',\delta}} f_\sharp \tau f^* = f_\sharp f^* \tau \xrightarrow{\beta'_f} \tau. \end{aligned}$$

where $g = f \circ f'' = f \circ f'$ is the projection $X \times_S X \to S$. Indeed the base change map $Ex^*_\sharp$ associated to the square Θ corresponds to the first identification in (4.2.8.a) and the adjunction map α_δ corresponds to the canonical map π.

Thus, we have to prove the preceding composite map is equal to the following one, obtained by adjunction from $\mathfrak{p}_f$:

$$\begin{aligned} f_\sharp f^* f_\sharp f^* = f_\sharp f^* f_\sharp f''_* \delta_* f^* \xrightarrow{Ex_{\sharp *}} f_\sharp f^* f_* f'_\sharp \delta_* f^* \\ \xrightarrow{\mathfrak{p}_{f',\delta}} f_\sharp f^* f_* \tau f^* = f_\sharp f^* f_* f^* \tau \xrightarrow{\alpha'_f} f_\sharp f^* \tau \xrightarrow{\beta'_f} \tau \end{aligned}$$

On can check after some easy cancellation that this amounts to prove the commutativity of the following diagram:

$$\begin{tikzcd}
f^* f_\sharp \arrow[r, equal] \arrow[d, "Ex_\sharp^*"'] & f^* f_\sharp f''_* \delta_* \arrow[r, "Ex_{\sharp *}"] & f^* f_* f''_\sharp \delta_* \arrow[d, "\alpha'_f"] \\
f'_\sharp f''^* \arrow[r, "\alpha_\delta"] & f'_\sharp \delta_* \delta^* f''^* \arrow[r, equal] & f'_\sharp \delta_* .
\end{tikzcd}$$

Using formula (4.2.2.a), we can divide this diagram into the following pieces:

$$\begin{tikzcd}
f^* f_\sharp \arrow[r, equal] \arrow[d, "Ex_\sharp^*"'] & f^* f_\sharp f''_* \delta_* \arrow[r, "\alpha_f"] \arrow[d, "Ex_\sharp^*"'] & f^* f_* f^* f_\sharp f''_* \delta_* \arrow[r, "Ex_\sharp^*"] & f^* f_* f'_\sharp f''^* f''_* \delta_* \arrow[r, "\alpha'_{f''}"] \arrow[d, "\alpha'_f"] & f^* f_* f''_\sharp \delta_* \arrow[d, "\alpha'_f"] \\
f'_\sharp f''^* \arrow[r, equal] \arrow[d, equal] & f'_\sharp f''^* f''_* \delta_* \arrow[rr, equal] \arrow[urr, "\alpha_f"] & \arrow[d, phantom, "(*)"] & f'_\sharp f''^* f''_* \delta_* \arrow[r, "\alpha'_{f''}"] & f'_\sharp \delta_* \arrow[d, equal] \\
f'_\sharp f''^* \arrow[rrrr, "\alpha_\delta"] & & {} & & f'_\sharp \delta_* .
\end{tikzcd}$$

Every part of this diagram is obviously commutative except for part $(*)$. As $f''\delta = 1$, the axioms of a 2-functors (for f^* and f_* say) implies that the unit map

$$f'_\sharp f''^* \xrightarrow{\alpha_{f''\delta}} f'_\sharp f''^* (f''\delta)_* (f''\delta)^*$$

is the canonical identification that we get using $1_* = 1$ and $1^* = 1$. We can consider the following diagram:

$$\begin{tikzcd}
f'_\sharp f''^* \arrow[rr, equal, "\alpha_{f''\delta}"] \arrow[d, equal] & & f'_\sharp f''^* (f''\delta)_* (f''\delta)^* \arrow[r, equal] \arrow[d, equal] & f'_\sharp f''^* f''_* \delta_* \arrow[dd, "\alpha'_{f''}"] \\
f'_\sharp f''^* \arrow[r, "\alpha_{f''}"] \arrow[d, equal] & f'_\sharp f''^* f''_\sharp f''^* \arrow[r, "\alpha_\delta"] \arrow[d, "\alpha'_{f''}"] & f'_\sharp f''^* (f''\delta)_* (f''\delta)^* \arrow[d, "\alpha'_{f''}"] & \\
f'_\sharp f''^* \arrow[r, equal] & f'_\sharp f''^* \arrow[r, "\alpha_\delta"] & f'_\sharp \delta_* \delta^* f''^* \arrow[r, equal] & f'_\sharp \delta_*
\end{tikzcd}$$

for which each part is obviously commutative. This concludes. □

This theorem will be generalized later on (see Corollary 4.3.2, point (3)). The important fact for the time being is the following corollary:

Corollary 4.2.9. *Under the hypothesis of Remark 4.2.2, if we assume that p is projective and smooth, the morphism $Ex_{\sharp *} : p_\sharp g_* \to f_* q_\sharp$ is an isomorphism.*

In fact, putting $\tau(K) = K(d)[2d]$ where d is the dimension of p, one checks easily that the following diagram is commutative:

$$\begin{tikzcd}
p_\sharp g_* \arrow[rr, "Ex_{\sharp *}"] \arrow[d, "\mathfrak{p}_p"'] & & f_* q_\sharp \arrow[d, "\mathfrak{p}_q"] \\
p_* \tau g_* & p_* g_* \tau \arrow[l, "Ex_\tau"'] \arrow[r, equal] & f_* q_* \tau
\end{tikzcd}$$

where we use formula (4.2.6.b) for the isomorphism Ex_τ.

4.3. **Localization.**

Theorem 4.3.1. *For any ring of positive characteristic R, the triangulated premotivic category $\mathrm{DM}_{\text{ét}}(-,R)$ satisfies the localization property (see Definition A.1.12).*

Proof. We will prove that condition (iii) of Proposition 2.3.4 is satisfied. Note that according to Proposition 4.1.1, i_* commutes with twists.[3] Thus it remains to prove that for any smooth S-scheme X, the canonical morphism

$$\epsilon_{X/S} : R_S^{tr}(X/X-X_Z) \to i_* R_Z^{tr}(X_Z)$$

is an isomorphism in $\mathrm{DM}_{\text{ét}}(S,R)$ (recall that $i_* = \mathbf{R}i_*$ according to Lemma 2.3.1).

Let us first consider the case where X is étale. Then according to Corollary 2.1.9, the sequence of sheaves with transfers

$$0 \to R_S^{tr}(X-X_Z) \xrightarrow{j_*} R_S^{tr}(X) \xrightarrow{i^*} i_* R_Z^{tr}(X_Z) \to 0 \tag{4.3.1.a}$$

is isomorphic after applying the functor γ_* to the sequence

$$0 \to R_S(X-X_Z) \xrightarrow{j_*} R_S(X) \xrightarrow{i^*} i_* R_Z(X_Z) \to 0.$$

This sequence of sheaves is obviously exact (we can easily check this on the fibres). As γ_* is conservative and exact, the sequence (4.3.1.a) is exact. Thus the canonical map:

$$R_S^{tr}(X/X-X_Z) := \mathrm{coker}(j_*) \to i_* R_Z^{tr}(X_Z)$$

is an isomorphism in $\mathrm{Sh}_{\text{ét}}^{tr}(X,R)$ and *a fortiori* in $\mathrm{DM}_{\text{ét}}(S,R)$.

We now turn to the general case. For any open cover $X = U \cup V$, we easily get the usual Mayer-Vietoris short exact sequence in $\mathrm{Sh}_{\text{ét}}^{tr}(S,R)$:

$$0 \to R_S^{tr}(U \cap V) \to R_S^{tr}(U) \oplus R_S^{tr}(V) \to R_S^{tr}(X) \to 0.$$

Thus the assertion is local on X for the Zariski topology. In particular, as X/S is smooth, we can assume there exists an étale map $X \to \mathbf{A}_S^n$. Therefore, by composing with any open immersion $\mathbf{A}_S^n \to \mathbf{P}_S^n$, we get an étale S-morphism $f : X \to \mathbf{P}_S^n$. Consider the following cartesian square:

$$\begin{tikzcd} \mathbf{P}_Z^n \arrow[r, "k"] \arrow[d, "q"'] & \mathbf{P}_S^n \arrow[d, "p"] \\ Z \arrow[r, "i"] & S, \end{tikzcd}$$

where p is the canonical projection. If we consider the notations of Paragraph 4.2.1 and Remark 4.2.2 relative to this square, then the following diagram

$$\begin{tikzcd} p_\sharp \arrow[rr, "p_\sharp(\alpha_k)"] \arrow[d, equal] & & p_\sharp k_* k^* \arrow[d, "Ex_{\sharp *}"] \\ p_\sharp \arrow[r, "\alpha_i"] & i_* i^* p_\sharp \arrow[r, "Ex_\sharp^*"] & i_* q_\sharp k^* \end{tikzcd}$$

is commutative – this can be easily checked using Formula (4.2.2.a).

---

[3] Essentially because it is true for its left adjoint i^*. This fact was already remarked at the beginning of the proof of Theorem 4.2.6.

If we apply the preceding commutative diagram to the object $R^{tr}_S(X/X-X_Z)$, we get the following commutative diagram in $\mathrm{DM}_{\text{ét}}(S,R)$:

$$\begin{tikzcd}
p_\sharp R^{tr}_{\mathbf{P}^n_S}(X/X-X_Z) \arrow[rr, "p_\sharp(\epsilon_{X/\mathbf{P}^n_S})"] \arrow[d, equal] & & p_\sharp k_* R^{tr}_{\mathbf{P}^n_Z}(X_Z) \arrow[d, "Ex_{\sharp *}"] \\
R^{tr}_S(X/X-X_Z) \arrow[r, "\epsilon_{X/S}"] & i_* q_\sharp R^{tr}_Z(X_Z) \arrow[r, equal] & i_* q_\sharp R^{tr}_{\mathbf{P}^n_Z}(X_Z)
\end{tikzcd}$$

The conclusion follows from the case treated above and from Corollary 4.2.9. □

As the premotivic triangulated category $\mathrm{DM}_{\text{ét}}(-,R)$ satisfies the stability property (Proposition 4.1.1) and the weak purity property (Theorem 4.2.6) the previous result allows to apply Theorem A.1.13 to $\mathrm{DM}_{\text{ét}}(-,R)$:

Corollary 4.3.2. *For any ring R of positive characteristic, the triangulated premotivic category $\mathrm{DM}_{\text{ét}}(-,R)$ satisfies Grothendieck's 6 functors formalism (Definition A.1.10).*

4.4. **Compatibility with direct image.**

4.4.1. According to Example A.1.3, the categories $\mathrm{D}(X_{\text{ét}},R)$ are the fibers of an Ét-premotivic triangulated category over *Sch*.

Note that the derived tensor product $\otimes^{\mathbf{L}}$ is essentially characterized by the property that for any étale X-schemes U and V, $R_X(U)\otimes^{\mathbf{L}}R_X(V)=R_X(U\times_X V)$ in $D(X_{\text{ét}},R)$.

Similarly, for any étale morphism $p:V\to X$, the operation $\mathbf{L}p_\sharp$ is characterized by the property that for any étale V-scheme W, $\mathbf{L}p_\sharp(R_V(W))=R_X(W)$.

4.4.2. In what follows, we will drop the letters $\mathbf{L}$ and $\mathbf{R}$ in front of derived functors to simplify notations.

Due to the properties of the functors involved in the construction of

$$\rho_!:\mathrm{D}(-_{\text{ét}},R)\to \mathrm{DM}^{eff}_{\text{ét}}(-,R)$$

we get the following compatibility properties:

1. $\rho_!$ is monoidal.
2. For any morphism $f:Y\to X$ of schemes, there exists a canonical isomorphism:
$$Ex(f^*,\rho_!):f^*\rho_!\to\rho_! f^*.$$
3. For any étale morphism $p:V\to X$, there exists a canonical isomorpism:
$$\rho_! p_\sharp\to p_\sharp\rho_!.$$

Assume that R is of positive characteristic n, and consider now a proper morphism $f:Y\to X$ between schemes whose residue characteristics are prime to n. Then, we can form the following natural transformation:

$$Ex(\rho_!,f_*):\rho_! f_*\xrightarrow{\alpha_f} f_*f^*\rho_! f_*\xrightarrow{Ex(f^*,\rho_!)} f_*\rho_! f^* f_*\xrightarrow{\alpha'_f} f_*\rho_!.$$

Proposition 4.4.3. *Using the assumptions and notations above, the map*

$$Ex(\rho_!,f_*):\rho_! f_*(K)\to f_*\rho_!(K)$$

is an isomorphism for any object K of $\mathrm{D}(Y_{\text{ét}},R)$.

Proof. Recall the triangulated category $\mathrm{DM}_{\text{ét}}(X,R) = \mathrm{DM}^{eff}_{\text{ét}}(X,R)$ is generated by objects of the form $R^{tr}_X(W) = p_\sharp(\mathbb{1}_W)$ where $p : W \to X$ is a smooth morphism. Thus, we have to prove that for any integer $n \in \mathbf{Z}$, the induced map:

$$\text{(4.4.3.a)}\quad \mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(p_\sharp(R_W)[n], \rho_! f_*(K)) \to \mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(p_\sharp(R_W)[n], f_*\rho_!(K)).$$

Consider the following cartesian square:

$$\begin{tikzcd} W' \arrow[r, "q"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ W \arrow[r, "p"] & X \end{tikzcd}$$

Then we get canonical isomorphisms

$$Ex^*_* : p^* f_* \to g_* q^*$$

both in $D(-_{\text{ét}},R)$ and in the premotivic triangulated category $\mathrm{DM}_{\text{ét}}(-,R)$, by the proper base change theorem – see Theorem 1.2.1 and respectively Corollary 4.3.2, Definition A.1.10(4).

On the other hand, the following diagram is commutative:

$$\begin{tikzcd} p^*\rho_! f_* \arrow[r, "Ex(\rho_!, f_*)"] \arrow[d, "Ex(p^*,\rho_!)"'] & p^* f_* \rho_! \arrow[d, "Ex^*_*"] \\ \rho_! p^* f_* \arrow[d, "Ex^*_*"'] & g_* q^* \rho_! \arrow[d, "Ex(q^*,\rho_!)"] \\ \rho_! g_* q^* \arrow[r, "Ex(\rho_!, g_*)"] & g_* \rho_! q^* \end{tikzcd}$$

Thus, using the adjunction $(p_\sharp, p^*)$ and replacing K by $g^*(K)[-n]$, we reduce to prove that the map (4.4.3.a) is an isomorphism for any complex K when $p = 1_X$ and $n = 0$. We have to prove that the map

$$Ex(\rho_!, f_*)_* : \mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(R_X, \rho_! f_*(K)) \to \mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(R_X, f_*\rho_!(K))$$

is an isomorphism.

But using the fact $\rho_!(R_X) = R_X$, Proposition 3.1.7, as well as the adjunction (f^*, f_*), the source and target of this map can be identified to $H^0_{\text{ét}}(Y,K)$ and this concludes. For the cautious reader, let us say more precisely that this follows from the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(R_X, f_*(K)) \arrow[r, "adj."] \arrow[d, "\rho_!"'] & \mathrm{Hom}(f^*(R_X), K) \arrow[d, "\rho_!"] & \\ \mathrm{Hom}(\rho_!(R_X), \rho_! f_*(K)) \arrow[d, equal] & \mathrm{Hom}(\rho_! f^*(R_X), \rho_!(K)) \arrow[r, "Ex(f^*,\rho_!)^*"] & \mathrm{Hom}(f^*\rho_!(R_X), \rho_!(K)) \arrow[d, "adj."] \\ \mathrm{Hom}(\rho_!(R_X), \rho_! f_*(K)) \arrow[rr, "Ex(\rho_!, f_*)_*"] & & \mathrm{Hom}(\rho_!(R_X), f_*\rho_!(K)). \end{tikzcd}$$

□

4.5. **The rigidity theorem.**

4.5.1. Given a scheme X, we denote by $\mathrm{D}^b_c(X_{\text{ét}},R)$ the full subcategory of $\mathrm{D}(X_{\text{ét}},R)$ which consists of complexes of étale sheaves with bounded constructible cohomology.

Lemma 4.5.2. *Using the notations above, for any scheme X, the category $\mathrm{D}^b_c(X_{\text{ét}},R)$ is the smallest triangulated thick subcategory of $D(X_{\text{ét}},R)$ generated by objects of the form $R_X(V)=\mathbf{L}p_\sharp(R_V)$ for $p:V\to X$ an étale morphism.*

As an obvious corollary, we obtain that $\mathrm{D}^b_c(X_{\text{ét}},R)$ is stable by the operations f^*, for any morphism f, by the operation $\mathbf{L}f_\sharp$ whenever f is étale, as well as by the operation $\otimes^{\mathbf{L}}$.

Recall from [CD12, Definition 1.4.9] the following definition, taken from Ayoub's Astérisques:

Definition 4.5.3. We define the category of *constructible étale motives* $\mathrm{DM}_{\text{ét},c}(X,R)$ as the thick triangulated subcategory of $\mathrm{DM}_{\text{ét}}(X,R)$ generated by objects of the form $R_X(Y)(n)$ for any smooth X-scheme Y and any integer $n\in\mathbf{Z}$.

We then have the following result (see [Ayo07, lemma 2.2.23] or [CD12, Proposition 4.2.13]), which uses Theorem 4.3.1:

Proposition 4.5.4. *If R is of positive characteristic, the category $\mathrm{DM}_{\text{ét},c}(X,R)$ is the thick triangulated subcategory of $\mathrm{DM}_{\text{ét}}(X,R)$ generated by objects of the form $f_*(R_Y)(n)$ for any projective morphism $f:Y\to X$ and any integer $n\in\mathbf{Z}$.*

The following theorem is a generalization of the rigidity theorem of Suslin and Voevodsky ([Voe96, 4.1.9] or [VSF00, chap. 5, 3.3.3]) when the base is of positive dimension:

Theorem 4.5.5. *Assume that R is a ring of positive characteristic n, and consider a noetherian $\mathbf{Z}[1/n]$-scheme X. Then the functor*

$$\rho_!:\mathrm{D}(X_{\text{ét}},R)\to\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)\simeq\mathrm{DM}_{\text{ét}}(X,R)$$

is an equivalence of symmetric monoidal triangulated categories, whose quasi-inverse is induced by the restriction functor on the small étale site. This equivalence of categories restricts to an equivalence at the level of constructible objects:

$$\mathrm{D}^b_c(X_{\text{ét}},R)\simeq\mathrm{DM}_{\text{ét},c}(X,R).$$

Proof. The fully faithfulness of the functor $\rho_!$ has been established in Proposition 3.1.7. According to Lemma 4.5.2, and points (1), (3), of Paragraph 4.4.2, we get that $\rho_!(\mathrm{D}^b_c(X_{\text{ét}},R))\subset\mathrm{DM}_{\text{ét},c}(X,R)$. On the other hand, using the fact that the direct image functors by proper morphisms preserve constructible coefficients (see [AGV73, Exposé XIV, 1.1]), together with Propositions 4.4.3 and 4.5.4, we get the converse inclusion. Furthermore, we know from Proposition 3.1.7 that the functor

$$\rho_!:\mathrm{D}(X_{\text{ét}},R)\to\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)$$

is fully faithful, and it is easy to see that it has a right adjoint

$$\rho^*:\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)\to\mathrm{D}(X_{\text{ét}},R)$$

defined by the restriction to the small étale site of X: $\rho^*(K)=K|_{X_{\text{ét}}}$. To prove that $\rho_!$ is an equivalence of categories, it is sufficient to prove that, for any object K of $\mathrm{DM}^{\mathit{eff}}_{\text{ét}}(X,R)$, the co-unit map

$$\rho_!\rho^*(K)\to K$$

is an isomorphism. For this, it is sufficient to prove that, for any smooth X-scheme V and any integer n, the map

$$\mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(R_X(V),\rho_!\rho^*(K)[n]) \to \mathrm{Hom}_{\mathrm{DM}^{eff}_{\text{ét}}(X,R)}(R_X(V),K[n])$$

is invertible. But we already know that $\mathrm{D}^b_c(X_{\text{ét}},R) \simeq \mathrm{DM}_{\text{ét},c}(X,R)$, so that $R_X(V) \simeq \rho_!(C)$ for some complex C in $\mathrm{D}^b_c(X_{\text{ét}},R)$. Therefore, it is sufficient to prove that the map

$$\rho^*\rho_!\rho^*(K) \to \rho^*(K)$$

is invertible, which follows easily from the fact that the functor $\rho_!$ is full faithful. □

We can extend these results in the case of p-torsion coefficients as follows:

Corollary 4.5.6. *Assume that R is of characteristic p^r for a prime p and an integer $r \geq 1$. Let X be any noetherian scheme, and $X[1/p] = X \times \mathrm{Spec}(\mathbf{Z}[1/p])$. Then there is a canonical equivalence of categories*

$$\mathrm{DM}_{\text{ét}}(X,R) \simeq \mathrm{D}(X[1/p]_{\text{ét}},R)\,.$$

Proof. This follows from Theorem 4.5.5 and from Proposition A.3.4. □

Corollary 4.5.7. *Under the assumptions of Theorem 4.5.5, for any complex of étale sheaves with transfers of R-modules C over X, the following conditions are equivalent:*

(i) *the complex C is $\mathbf{A}^1$-local;*
(ii) *for any integer n, the étale sheaf $H^n(C)$ (seen as a complex concentrated in degree zero) is $\mathbf{A}^1$-local;*
(iii) *the map $\rho_!\rho^*C \to C$ is a quasi-isomorphism of complexes of étale sheaves;*
(iv) *for any integer n, the map $\rho_!\rho^*H^n(C) \to H^n(C)$ is invertible.*

Proof. The equivalence between conditions (i) and (iii) follows immediately from Theorem 4.5.5, from which we deduce the equivalence between conditions (ii) and (iv). The equivalence between conditions (iii) and (iv) comes from the fact that both $\rho_!$ and ρ^* are exact functors. □

4.6. **Absolute purity with torsion coefficients.**

Theorem 4.6.1. *For any ring of positive characteristic, the triangulated premotivic category $\mathrm{DM}_{\text{ét}}(-,R)$ satisfies the absolute purity property (Definition A.2.9).*

This means in particular that for any closed immersion $i: Z \to S$ between regular schemes, one has a canonical isomorphism in $\mathrm{DM}_{\text{ét}}(S,R)$:

$$\eta_X(Z): R_Z \to i^!(R_S)(c)[2c].$$

Proof. For any closed immersion $i: Z \to S$, we define a complex of R-modules using the dg-enrichement of $\mathrm{DM}_{\text{ét}}(S,R)$:

$$\mathbf{R}\Gamma_Z(X) = \mathbf{R}Hom(i_*(R_Z),R_S).$$

This complex is contravariant in (X,Z) – see A.2.1 for morphisms of closed pairs. We have to prove that whenever S and R are regular, the maps induced by the deformation diagram (A.2.7.a),

$$\mathbf{R}\Gamma_Z(X) \xleftarrow{d_1^*} \mathbf{R}\Gamma_{\mathbf{A}^1_Z}(D_Z X) \xrightarrow{d_0^*} \mathbf{R}\Gamma_Z(N_Z X)$$

are quasi-isomorphism. We may assume that $R = \mathbf{Z}/n\mathbf{Z}$ for some natural number $n > 0$. By a simple devissage, we may as well assume that n is a power of some prime p. By virtue of Corolary 4.5.6, we see that all this is a reformulation of the analogous property in the setting of classical étale cohomology, with coefficients prime to the residue characteristics. We conclude with Gabber's absolute purity theorem (see [Fuj02]). □

5. h-MOTIVES AND ℓ-ADIC REALISATION

In this section, we fix a ring Λ such that $\mathbf{Z} \subset \Lambda \subset \mathbf{Q}$ and consider a Λ-algebra R. We let *Sch* be the category of noetherian schemes.

In addition, for any base scheme S in *Sch*, we let $\mathscr{S}^{ft}_S$ be the category of S-schemes of finite type.

5.1. h-motives.

5.1.1. Recall that Voevodsky has defined the h-topology on the category of noetherian schemes as the topology whose covers are the universal topological epimorphisms; see [Voe96, 3.1.2]. Given a noetherian scheme S as well as a ring R, we will denote by $\mathrm{Sh}_{\mathrm{h}}(S,R)$ the category of h-sheaves of R-modules on the category $\mathscr{S}^{ft}_S$. Given any S-scheme X of finite type, we will denote by $\underline{R}^{\mathrm{h}}_S(X)$ the free h-scheaf or R-modules represented by X. As proved in [CD12, Ex. 5.1.4], the *Sch*-fibered category $\mathrm{Sh}_{\mathrm{h}}(-,R)$ is an abelian $\mathscr{S}^{ft}$-premotivic category in the sense of Definition A.1.1.

The following definition, although using the theory of [CD12] for the existence of derived functors, follows the original idea of Voevodsky in [Voe96]:

Definition 5.1.2. Using the notations above, we define the $\mathscr{S}^{ft}$-premotivic *big category of effective* h-*motives* (resp. of h-*motives*) with R-linear coefficients

$$\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(-,R) \quad (\text{resp. } \underline{\mathrm{DM}}_h(-,R))$$

as the $\mathbf{A}^1$-derived category (resp. stable $\mathbf{A}^1$-derived category) associated with the *Sch*-fibred category $\mathrm{Sh}_{\mathrm{h}}(-,R)$.

In other words, the triangulated monoidal category $\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$ is the $\mathbf{A}^1$-localisation of the derived category $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R))$; this is precisely the original definition of Voevodsky [Voe96, sec. 4]. This category is completely analogous to the case of the étale topology (2.2.4). Similarly, the category $\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$ is obtained from $\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$ by $\otimes$-inverting the Tate h-motive in the sense of model categories. We get functors as in (2.2.4.a):

$$\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R)) \xrightarrow{\pi_{\mathbf{A}^1}} \underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R) \xrightarrow{\Sigma^\infty} \underline{\mathrm{DM}}_{\mathrm{h}}(S,R). \tag{5.1.2.a}$$

Note however that the category $\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$ ($\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$) is generated by objects of the form $\underline{R}^{\mathrm{h}}_S(X)$ ($\Sigma^\infty\underline{R}^{\mathrm{h}}_S(X)(n)$) for any S-scheme of finite type X (for any S-scheme of finite type X and any integer $n \in, \mathbf{Z}$, respectively). These categories are too big to satisfy the 6 functors formalism (the drawback is about the localization property with respect to closed immersion, which means that there is no good theory of support).

This is why we introduce the following definition (following [CD12, Ex. 5.3.31]).

Definition 5.1.3. The category of *effective* h-*motives* (resp. of h-*motives*)

$$\mathrm{DM}^{eff}_{\mathrm{h}}(X,R) \quad (\text{resp. } \mathrm{DM}_{\mathrm{h}}(X,R))$$

is the smallest full subcategory of $\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$ (resp. of $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$) closed under arbitrary small sums and containing the objects of the form $\underline{R}^{\mathrm{h}}_S(X)$ (resp. $\Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)(n)$) for X/S smooth (resp. for X/S smooth and $n \in \mathbf{Z}$).

The category of *constructible effective* (resp. of *constructible*) h-*motives*

$$\mathrm{DM}^{eff}_{\mathrm{h},c}(X,R) \quad (\text{resp. } \mathrm{DM}_{\mathrm{h},c}(X,R))$$

is the thick triangulated subcategory of $\mathrm{DM}^{eff}_{\mathrm{h}}(S,R)$ (resp. $\mathrm{DM}_{\mathrm{h}}(X,R)$) generated by objects of the form $\underline{R}^{\mathrm{h}}_S(X)$(resp. $\Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)(n)$) for X/S smooth (resp. for X/S smooth and $n \in \mathbf{Z}$).

We will sometimes simplify the notations and write $R(X) = \Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)$, as an object of $\mathrm{DM}_{\mathrm{h}}(X,R)$ (for a smooth S-scheme X).

It is obvious that the subcategory $\mathrm{DM}_{\mathrm{h}}(-,R)$ is stable by the operations f^* for any morphism f, by the operation $f_\sharp$ for any *smooth* morphism f, and by the operation $\otimes^{\mathbf{L}}$. The Brown representability theorem implies that the inclusion functor $\nu_\sharp$ admits a right adjoint ν^*, so that $\mathrm{DM}_{\mathrm{h}}(-,R)$ is in fact a premotivic triangulated category, and we get an enlargement of premotivic triangulated category:

$$\nu_\sharp : \mathrm{DM}_{\mathrm{h}}(X,R) \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{h}}(S,R) : \nu^* \tag{5.1.3.a}$$

– see [CD12, Ex. 5.3.31(2)]. More precisely, for any morphism of schemes $f : X \to Y$, the functor

$$\mathbf{L}f^* : \mathrm{DM}_{\mathrm{h}}(Y,R) \to \mathrm{DM}_{\mathrm{h}}(X,R)$$

admits a right adjoint

$$\mathbf{R}f_* : \mathrm{DM}_{\mathrm{h}}(X,R) \to \mathrm{DM}_{\mathrm{h}}(Y,R)$$

defined by the formula

$$\mathbf{R}f_*(M) = \nu^*(\mathbf{R}f_*(\nu_\sharp(M))).$$

Similarly, the (derived) internal Hom of $\mathrm{DM}_{\mathrm{h}}(X,R)$ is defined by the formula

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) = \nu^*(\mathbf{R}\underline{\mathrm{Hom}}_R(\nu_\sharp(M),\nu_\sharp(N))).$$

We will sometimes write $\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) = \mathbf{R}\underline{\mathrm{Hom}}(M,N)$ when the coefficients are understood from the context. Also, when it is clear that we work with derived functors only, it might happen that we drop the thick letters $\mathbf{L}$ and $\mathbf{R}$ from the notations. The unit object of the monoidal category $\mathrm{DM}_{\mathrm{h}}(X,R)$ will witten $\mathbb{1}_X$ or R_X, depending on the emphasis we want to put on the coefficients.

Remark 5.1.4. The category $\underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(X,\mathbf{Z})$ is nothing else than the category introduced by Voevodsky in [Voe96] under the notation *DM*(*S*). The fact these categories must be some version of étale motives is clearly envisionned in *loc. cit.*

5.2. h-**descent for torsion étale sheaves.**

5.2.1. Given any noetherian scheme S and any ring R, proceeding as in 3.1.1, there is an exact fully faithful embedding of the category $\mathrm{Sh}(S_{\text{ét}},R)$ in the category of étale sheaves of R-modules over the big étale site of S-schemes of finite type. Composing this embedding with the h-sheafification functor leads to an exact functor

$$\alpha^* : \mathrm{Sh}(S_{\text{ét}},R) \to \mathrm{Sh}_{\mathrm{h}}(S,R)\,, \quad F \mapsto \alpha^*(F) = F_h. \tag{5.2.1.a}$$

This functor has a right adjoint

$$\alpha_* : \mathrm{Sh}_{\mathrm{h}}(S,R) \to \mathrm{Sh}(S_{\text{ét}},R). \tag{5.2.1.b}$$

which defined by $\alpha_*(F) = F|_{S_{\text{ét}}}$. The functor (5.2.1.a) induces a functor

$$\alpha^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R))\,. \tag{5.2.1.c}$$

which has a right adjoint

$$\mathbf{R}\alpha_* : \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R)) \to \mathrm{D}(S_{\text{ét}}, R)\,. \tag{5.2.1.d}$$

Lemma 5.2.2. *For any ring R and any noetherian scheme S, the derived restriction functor* (5.2.1.d) *preserves small sums.*

Proof. Let us prove first the lemma in the case where S is of finite Krull dimension as well as of finite étale cohomological dimension. Then any S-scheme of finite type has the same property. Moreover, by virtue of a theorem of Goodwillie and Lichtenbaum [GL01], any S-scheme of finite type has finite h-cohomological dimension as well. For a complex C of h-sheaves of R-modules over S, the sheaf cohomology $H^i(\mathbf{R}\alpha_*(C))$ is the étale sheaf associated to the presheaf

$$V \mapsto H^i_{\mathrm{h}}(V, C)\,.$$

It follows from Proposition 1.1.9 that the functors $H^i_{\mathrm{h}}(V, -)$ preserve small sums, which implies that the functor $\mathbf{R}\alpha_*$ has the same property.

We now can deal with the general case as follows. Let ξ be a geometric point of S, and write $u : S_\xi \to S$ for the canonical map from the strict henselization of S at ξ. We then have pullback functors

$$u^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(S_{\xi,\text{ét}}, R) \text{ and } u^* : \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R)) \to \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S_\xi, R))\,.$$

The family of functors u^* form a conservative family of functors which commutes with sums (when ξ runs over all geometric points of S). Therefore, it is sufficient to prove that the functor $u^*\mathbf{R}\alpha_*$ commutes with sums. Let V be an affine étale scheme over S_ξ. There exists a projective system of étale S-chemes $\{V_i\}$ with affine transition maps such that $V = \varprojlim_i V_i$. Note that S_ξ is of finite étale cohomological dimension (see Gabber's Theorem 1.1.5), so that, by virtue of Lemma 1.1.12, for any complex of sheaves of R-modules K over $S_{\text{ét}}$, one has

$$\varinjlim_i H^n_{\text{ét}}(V_i, K) \simeq H^n_{\text{ét}}(V, u^*(K))\,.$$

Similarly, applying Lemma 1.1.12 to the h-sites, for any complex of h-sheaves of R-modules L over S, we have

$$\varinjlim_i H^n_{\mathrm{h}}(V_i, L) \simeq H^n_{\mathrm{h}}(V, u^*(L))\,.$$

Note that, for any étale map $w : W \to S$, the natural map $w^*\mathbf{R}\alpha_*(C) \to \mathbf{R}\alpha_* w^*(C)$ is invertible. Therefore, for any complex of h-sheaves of R-modules C over S, we have natural isomorphisms

$$\begin{aligned}
H^n_{\text{ét}}(V, u^*\mathbf{R}\alpha_*(C)) &\simeq \varinjlim_i H^n_{\text{ét}}(V_i, \mathbf{R}\alpha_*(C)) \\
&\simeq \varinjlim_i H^n_{\mathrm{h}}(V_i, C)) \\
&\simeq H^n_{\mathrm{h}}(V, u^*(C)) \\
&\simeq H^n_{\text{ét}}(V, \mathbf{R}\alpha_* u^*(C))\,.
\end{aligned}$$

In other words, the natural map $u^*\mathbf{R}\alpha_* \to \mathbf{R}\alpha_* u^*$ is invertible, and as we already know that the functor $\mathbf{R}\alpha_*$ commutes with small sums over S_ξ, this achieves the proof of the lemma. □

Proposition 5.2.3. *Let R be a ring of positive characteristic, and S be a noetherian scheme. The functor* (5.2.1.c) *is fully faithful. In other words, for any complex C of sheaves of R-modules over $S_{\text{ét}}$, and for any morphism of finite type $f : X \to S$, the natural map*

$$H^i_{\text{ét}}(X, f^*C) \to H^i_{\mathrm{h}}(X, \alpha^*C)$$

is invertible for any integer i.

Proof. We must prove that, for any complex of sheaves of R-modules C over $S_{\text{ét}}$, the natural map

$$C \to \mathbf{R}\alpha_*\mathbf{L}\alpha^*(C)$$

is invertible in $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S,R))$. The functor $\mathbf{R}\alpha_*$ preserves small sums (Lemma 5.2.2). Therefore, it is sufficient restrict ourselves to the case of bounded complexes. Then, by virtue of [AGV73, Exposé Vbis, 3.3.3], it is sufficient to prove that any h-cover is a morphism of universal cohomological 1-descent (with respect to the fibred category of étale sheaves of R-modules). The h-topology is the minimal Grothendieck topology generated by open coverings as well as by coverings of shape $\{p : Y \to X\}$ with p proper and surjective; see [Voe96, 1.3.9] in the context of excellent schemes, and [Ryd10, 8.4] in general. We know that the class of morphisms of universal cohomological 1-descent form a pretopology on the category of schemes; see [AGV73, Exposé Vbis, 3.3.2]. To conclude the proof, it is thus sufficient to note that any étale surjective morphism (any proper surjective morphism, respectively) is a morphism of universal cohomological 1-descent; see [AGV73, Exposé Vbis, 4.3.5 & 4.3.2]. □

5.3. **Basic change of coefficients.**

5.3.1. Let R' be an R-algebra and S be a base scheme. We associate to R'/R the classical adjunction:

$$\rho^* : \mathrm{Sh}_h(S,R) \leftrightarrows \mathrm{Sh}_h(S,R') : \rho_* \tag{5.3.1.a}$$

such that $\rho^*(F)$ is the h-sheaf associated with the presheaf $X \mapsto F(X) \otimes_R R'$. The functor ρ_* is faithful, exact and commutes with arbitrary diret sums. Note also the formula:

$$\rho_*\rho^*(F) = F \otimes_R R' \tag{5.3.1.b}$$

where R' is seen as the constant h-sheaf associated with the R-module R'.

Note that the adjunction(5.3.1.a) is an adjunction of $\mathscr{S}^{ft}$-premotivic abelian categories. As such, it can be derived and induces a $\mathscr{S}^{ft}$-premotivic adjunction:

$$\mathbf{L}\rho^* : \underline{\mathrm{DM}}_h(-,R) \rightleftarrows \underline{\mathrm{DM}}_h(-,R') : \mathbf{R}\rho_*$$

which restricts, according to Definition 5.1.3, to a premotivic adjunction

$$\mathbf{L}\rho^* : \mathrm{DM}_h(-,R) \rightleftarrows \mathrm{DM}_h(-,R') : \mathbf{R}\rho_*. \tag{5.3.1.c}$$

Recall that the stable category of h-motives over S is a localization of the derived category of symmetric Tate spectra of h-sheaves over S.[4] Here we will simply denote this category by $\mathrm{Sp}_h(S,R)$ and call its object *s*pectra. The adjuntion (5.3.1.a) can be extended to an adjunction of $\mathscr{S}^{ft}$-premotivic abelian categories:

$$\rho^* : \mathrm{Sp}_h(-,R) \rightleftarrows \mathrm{Sp}_h(-,R') : \rho_*. \tag{5.3.1.d}$$

[4]See [CD12], Definition 5.3.16 for symmetric Tate spectra and Definition 5.3.22 for the stable $\mathbf{A}^1$-derived category.

Again, ρ_* is faithful, exact and commutes with arbitrary sums. Note that the model category structure on $\mathrm{Sp}_h(-,R')$ is a particular instance of a general construction (see [CD12, 7.2.1 and Theorem 7.2.2]), from which we immediately get the following useful result (which is not dificult to prove directly though):

Lemma 5.3.2. *The functor $\rho_* : \mathrm{C}(\mathrm{Sp}_h(S,R')) \to \mathrm{C}(\mathrm{Sp}_h(S,R))$ preserves and detects stable weak $\mathbf{A}^1$-equivalences.*

As a corollary, we get:

Proposition 5.3.3. *Consider the notations of Paragraph 5.3.1. The functors $\mathbf{R}\rho_* = \rho_*$ is conservative and admits a right adjoint:*

$$\rho^! : \underline{\mathrm{DM}}_h(S,R) \to \underline{\mathrm{DM}}_h(S,R').$$

For any h-motive M over S, the following computations hold:

$$\rho_*\mathbf{L}\rho^*(M) = M \otimes^{\mathbf{L}}_R R',$$
$$\rho_*\rho^!(M) = \mathbf{R}\underline{\mathrm{Hom}}_R(R',M).$$

5.3.4. We consider the particular case of the discussion above when $R = \mathbf{Z}$ and $R' = \mathbf{Z}/n\mathbf{Z}$ for a positive integer n. For any h-motive M over S, we put:

$$M/n := M \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z}. \tag{5.3.4.a}$$

Then the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z} \to 0$$

induces a canonical distinguished triangle in $\mathrm{DM}_h(S,\mathbf{Z})$:

$$M \xrightarrow{n} M \longrightarrow M/n \longrightarrow . \tag{5.3.4.b}$$

In the next statement, we will use the fact that $\underline{\mathrm{DM}}_h(S,R)$ is a dg-category (see [CD12, Rem. 5.1.19]). We denote the enriched Hom by $\mathbf{R}\mathrm{Hom}$.

Proposition 5.3.5. *Consider the previous notations. Let S be a scheme and $f : X \to S$ be a morphism of Sch, M and N be h-motives over X. Then the natural exchange transformations:*

$$\begin{array}{lrcl} (1) & \mathbf{R}f_*(N)/n & \longrightarrow & \mathbf{R}f_*(N/n), \\ (2) & \mathbf{R}\underline{\mathrm{Hom}}(M,N)/n & \longrightarrow & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(M/n,N/n), \\ (3) & \mathbf{R}\mathrm{Hom}(M,N)/n & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}/n\mathbf{Z}}(M/n,N/n), \end{array}$$

are isomorphisms.

Proof. In each case, this follows from the distinguished triangle (5.3.4.b) – or its analog in the derived category of abelian groups. □

5.3.6. Next we consider the case of $\mathbf{Q}$-localisation.

Proposition 5.3.7. *Let S be a quasi-excellent scheme dimension d. Then S is of cohomological dimension $\leq d$ for $\mathbf{Q}$-linear coefficients with respect to the* h*-topology. In particular, for any complex of* h*-sheaves K over S, and for any S-scheme of finite type, we have a canonical isomorphism*

$$H^0_{\mathrm{h}}(X,K) \otimes \mathbf{Q} \simeq H^0_{\mathrm{h}}(X,K \otimes \mathbf{Q}).$$

Proof. It readily follows from [CD12, Th. 3.3.25, 3.3.30] that the cohomology of a $\mathbf{Q}$-linear h-sheaf with respect to the h-topology coincides with its analogue for the cdh-topology. The first assertion thus follows from the fact that the cdh-cohomological dimension is bounded by the topological dimension; see [SV00a, Theorem 12.5]. The last assertion of the proposition is then a direct application of Lemma 1.1.10. □

As an immediate corollary, we get:

Corollary 5.3.8. *For any quasi-excellent scheme S of finite dimension, tensoring by $\mathbf{Q}$ preserves fibrant symmetric Tate spectra. Furthermore, for any S-scheme of finite type X, and for any object M of $\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$, we have*

$$\mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)}(\underline{R}^{\mathrm{h}}_S(X),M)\otimes\mathbf{Q}\simeq\mathrm{Hom}_{\underline{\mathrm{DM}}(S,R)}(\underline{R}^{\mathrm{h}}_S(X),N\otimes\mathbf{Q}).$$

Proof. The previous proposition shows that tensoring with $\mathbf{Q}$ preserves the property of cohomological h-descent, while it obviously preserves the properties of being homotopy invariant and of being an Ω-spectrum. This proves the first assertion. The second one, is a direct translation of the first. □

Corollary 5.3.9. *Consider a quasi-excellent scheme S of finite dimension and any ring R. For any objects M and N of $\mathrm{DM}_{\mathrm{h}}(S,R)$, if M is constructible, then*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S,R)}(M,N)\otimes\mathbf{Q}\simeq\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S,R)}(M,N\otimes\mathbf{Q}).$$

Proof. It is equivalent to show that the functor

$$\nu^*:\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)\to\mathrm{DM}_{\mathrm{h}}(S,R)$$

commutes with $\mathbf{Q}$-linearization (where, for an object E of $\mathrm{DM}_{\mathrm{h}}(S,R)$, one defines $E\otimes\mathbf{Q}=\nu^*(\nu_\sharp(E)\otimes\mathbf{Q})$). Let M be any object of $\underline{\mathrm{DM}}_{\mathrm{h}}(S)$, and X be a smooth separated S-scheme of finite type. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X),\nu^*(M)\otimes\mathbf{Q}) &\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X),\nu_\sharp(\nu^*(M))\otimes\mathbf{Q})\\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X),\nu_\sharp(\nu^*(M)))\otimes\mathbf{Q}\\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X),\nu^*(M))\otimes\mathbf{Q}\\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X),M)\otimes\mathbf{Q}\\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S,\mathbf{Q})}(\mathbf{Q}(X),M\otimes\mathbf{Q})\\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Q})}(\mathbf{Q}(X),\nu^*(M\otimes\mathbf{Q}))\\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X),\nu^*(M\otimes\mathbf{Q}))
\end{aligned}$$

As both functors $\nu_\sharp$ and ν^* preserve Tate twists, this implies that the canonical map $\nu^*(M)\otimes\mathbf{Q}\to\nu^*(M\otimes\mathbf{Q})$ is invertible for any M. □

Remark 5.3.10. This corollary says in particular that the category $\mathrm{DM}_{\mathrm{h},c}(S,R\otimes\mathbf{Q})$ of constructible h-motives with $R\otimes\mathbf{Q}$-coefficients is the pseudo-abelian enveloppe of the *naive* $\mathbf{Q}$-localisation of the triangulated category $\mathrm{DM}_{\mathrm{h},c}(S,R)$. This is not an obvious fact as the category $\mathrm{DM}_{\mathrm{h}}(S,R)$ is not compactly generated for general base schemes S.

As a corollary, we get the following analog of Proposition 5.3.5:

Corollary 5.3.11. *Let S be a scheme and $f : X \to S$ be a morphism of finite type, M and N be h-motives with R-coefficients over X, with M constructible. Then the natural exchange transformations below are isomorphisms:*

$$\begin{array}{lrcl} (1) & \mathbf{R}f_*(N)\otimes\mathbf{Q} & \longrightarrow & \mathbf{R}f_*(N\otimes\mathbf{Q}), \\ (2) & \mathbf{R}\underline{\mathrm{Hom}}_R(M,N)\otimes\mathbf{Q} & \longrightarrow & \mathbf{R}\underline{\mathrm{Hom}}_{R\otimes\mathbf{Q}}(M\otimes\mathbf{Q},N\otimes\mathbf{Q}), \\ (3) & \mathbf{R}\mathrm{Hom}(M,N)\otimes\mathbf{Q} & \longrightarrow & \mathbf{R}\mathrm{Hom}(M\otimes\mathbf{Q},N\otimes\mathbf{Q}). \end{array}$$

Proof. It is sufficient to check this after applying the functor $\mathrm{Hom}_{\mathrm{DM}_h(S,R)}(P,-)$ for any constructible h-motive P with coefficients in R. Then the result follows again from Corollary 5.3.9. □

As a notable application of the results proved so far, we get the following proposition:

Proposition 5.3.12. *Let $\mathscr{P}$ be the set of prime integers and S be a scheme. Then the family of functors:*

$$\rho^* : \mathrm{DM}_h(S,R) \to \mathrm{DM}_h(S,R_{\mathbf{Q}}),$$
$$\rho_p^* : \mathrm{DM}_h(S,R) \to \mathrm{DM}_h(S,R/p), p \in \mathscr{P},$$

defined above is conservative.

Proof. Let K be an h-motive over S with coefficients in R such that $\rho^*(K)=0$ and $\rho_p^*(K)=0$ for all $p\in\mathscr{P}$.

It is sufficient to prove that for any constructible h-motive M, $\mathrm{Hom}(M,K)=0$. Given any prime p, the fact $\rho_p^*(K)=0$ together with the distinguished triangle (5.3.4.b) implies that the abelian group $\mathrm{Hom}(M,K)$ is uniquely p-divisible. As this is true for any prime p, we get: $\mathrm{Hom}(M,K)=\mathrm{Hom}(M,K)\otimes\mathbf{Q}$. But, as M is constructible, Corollary 5.3.9 implies the later group is isomorphic to $\mathrm{Hom}(\rho^*(M),\rho^*(K))$ which is zero by assumption on K. □

5.4. **Comparison theorem.**

5.4.1. Recall from [CD12, Par. 14.2.20] the category $\mathrm{DM}_{\mathrm{B}}(X,R)$ of Beilinson motives. The following theorem was proved in [CD12, Th. 16.1.2]:

Theorem 5.4.2. *Given any any quasi-excellent scheme X of finite Krull dimension, there exists a canonical equivalence of symmetric monoidal triangulated categories:*

$$\mathrm{DM}_{\mathrm{B}}(X) \simeq \mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q}).$$

This means, in particular, that, if X is regular, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})}(\mathbf{Q}_X,\mathbf{Q}_X(p)[q]) \simeq Gr_\gamma^p K_{2p-q}(X)\otimes\mathbf{Q},$$

where the second term stands for the graded pieces of algebraic K-theory with respect to the γ-filtration (by virtue of a theorem of Voevodsky, the regularity assumption can be dropped if we replace K-theory by its homotopy invariant version in the sense of Weibel; see [CD12, 14.1.1] *and* [Cis13]*).*

5.4.3. Recall Λ is a sub-ring of $\mathbf{Q}$ and R is a Λ-algebra. As it appears already in Paragraph 2.1.1, finite S-correspondences with coefficients in Λ are defined for separated S-schemes of finite type. According to [CD12, Def. 9.1.8], they define a category which we will denote by $\mathscr{S}^{cor}_{\Lambda,S}$.

Given any S-scheme X, we denote by $\underline{R}_S^{tr}(X)$ the presheaf of R-modules on $\mathscr{S}^{cor}_{\Lambda,S}$ represented by X. Moreover the graph functor induces a canonical morphism of presheaves on $\mathscr{S}^{ft}_S$:

$$\underline{R}_S(X) \to \underline{R}_S^{tr}(X). \tag{5.4.3.a}$$

Recall the following result of Suslin and Voevodsky (see [VSF00, 4.2.7 and 4.2.12]).

Proposition 5.4.4. *The map* (5.4.3.a) *induces an isomorphism after* h-*sheafification. Furthermore, if S is a noetherian $\mathbf{Z}[1/n]$-scheme of finite dimension and if any integer prime to n is invertible in R, then, for any S-scheme X of finite type, the presheaf $\underline{R}_S^{tr}(X)$ is a* qfh-*sheaf, and the morphism* (5.4.3.a) *induces an isomorphism of* qfh-*sheaves:*

$$\underline{R}_S^{qfh}(X) \to \underline{R}_S^{tr}(X).$$

This implies in particular that any h-sheaf F over S defines by restriction an étale sheaf with transfers $\psi^*(F)$, on Sm^{cor}_S (without any restriction on the characteristic). This gives a canonical functor:

$$\psi^* : \mathrm{Sh}_{\mathrm{h}}(S,R) \to \mathrm{Sh}^{tr}_{\text{ét}}(S,R)$$

which preserves small limits as well as small filtering colimits. Using the argument of the prooof of [CD12, Theorem 10.5.14], one can show this functor admits a left adjoint $\psi_!$ uniquely defined by the property that $\psi_!(R_S^{tr}(X)) = \underline{R}_S^{\mathrm{h}}(X)$ for any smooth S-scheme X.

We have defined an adjunction of premotivic categories over $Sch[1/n]$:

$$\psi_! : \mathrm{Sh}^{tr}_{\text{ét}}(-,R) \rightleftarrows \mathrm{Sh}_{\mathrm{h}}(-,R) : \psi^*.$$

According to [CD12, 5.2.19], these functors can be derived and induce an adjunction of premotivic categories over $Sch[1/n]$:

$$\mathbf{L}\psi_! : \mathrm{DM}^{eff}_{\text{ét}}(-,R) \rightleftarrows \underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(-,R) : \mathbf{R}\psi^*.$$

As a consequence of the rigidity theorem 4.5.5 and of the cohomological h-descent property for étale topology 5.2.3, we get:

Theorem 5.4.5. *Assume that the ring R is of positive characteristic. For any noetherian scheme of finite dimension S, the functor $\mathbf{L}\psi_! : \mathrm{DM}^{eff}_{\text{ét}}(S,R) \to \underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$ is fully faithful and induces an equivalence of triangulated categories*

$$\mathrm{DM}^{eff}_{\text{ét}}(S,R) \xrightarrow{\sim} \mathrm{DM}^{eff}_{\mathrm{h}}(S,R) \xrightarrow{\sim} \mathrm{DM}_{\mathrm{h}}(S,R).$$

Proof. The equivalence $\mathrm{DM}^{eff}_{\text{ét}}(S,R) \simeq \mathrm{DM}^{eff}_{\mathrm{h}}(S,R)$ follows from the first assertion: the essential image of $\mathbf{L}\psi_!$ is obviously included in $\mathrm{DM}^{eff}_{h}(S,R)$ because $\mathbf{L}\psi_!(R_S^{tr}(X)) = \underline{R}_S^h(X)$ for any smooth S-scheme. Let n be the characteristic of R. As R is a $\mathbf{Z}/n\mathbf{Z}$-algebra, to prove that the functor $\mathbf{L}\psi_!$ is fully faithful, it is sufficient to consider the case where $R = \mathbf{Z}/n\mathbf{Z}$. Decomposing n into its prime factors, we are thus reduced to prove that $\mathbf{L}\psi_!$ is fully faithful in the case where $n = p^a$ with p a prime and $a \geq 1$. Furthermore, by virtue of Proposition A.3.4, we may assume that n is invertible in the residue fields of S. In this case, we know that the composite functor

$$\tilde{\rho}_! : \mathrm{D}(S_{\text{ét}},R) \xrightarrow{\rho_!} \mathrm{DM}^{eff}_{\text{ét}}(S,R) \xrightarrow{\mathbf{L}\psi_!} \underline{\mathrm{DM}}^{eff}_{\mathrm{h}}(S,R)$$

is fully faithful (Proposition 5.2.3) and that the functor $\rho_!$ is an equivalence of categories (by the rigidity theorem 4.5.5). This obviously implies that the functor $\mathbf{L}\psi_!$ is fully faithful.

For the last equivalence, we simply notice that, for any ring of positive characteristic R, the premotivic triangulated category $\underline{\mathrm{DM}}_{\mathrm{h}}^{eff}(S,R)$ satisfies the stability property with respect to the Tate object $R(1)$, so that we get a canonical equivalence of categories

$$\underline{\mathrm{DM}}_{\mathrm{h}}^{eff}(S,R) \simeq \underline{\mathrm{DM}}_{\mathrm{h}}(S,R).$$

This induce an equivalence of categories $\mathrm{DM}_{\mathrm{h}}^{eff}(S,R) \simeq \mathrm{DM}_{\mathrm{h}}(S,R)$. □

Using the preceding theorem, together with Theorem 4.5.5 and Proposition A.3.4, we finally get:

Corollary 5.4.6. *Assume R is a ring of positive characteristic $n = p^a$, p being a prime. Then for any noetherian scheme X of finite dimension, with p invertible in the residue fields of X, there are canonical equivalences of triangulated monoidal categories*

$$D(X_{\text{ét}},R) \simeq \mathrm{DM}_{\mathrm{h}}(X,R),$$

which restricts to equivalences of triangulated monoidal categories:

$$D^b_c(X_{\text{ét}},R) \simeq \mathrm{DM}_{\mathrm{h},c}(X,R).$$

Moreover, these equivalences of categories induce an equivalence of premotivic triangulated categories over Sch:

$$D\big((-)_{\text{ét}},R\big) \simeq \mathrm{DM}_{\mathrm{h}}(-,R).$$

Recall the last statement is equivalent to assert that these equivalences are compatible with all of the 6 operations.

Recall from [CD12, 5.3.31] the triangulated category $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X,R) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(X,R))$, obtained as the stabilisation of the $\mathbf{A}^1$-derived category of étale sheaves on the smooth-étale site of X. The category $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X,R)$ is taken in Ayoub's paper [Ayo] as the model for étale motives.

Corollary 5.4.7. *Let X be a quasi-excellent noetherian scheme of finite dimension. Assume either that all the residue fields of X are of characteristice zero, or that 2 is invertible in R. Then the canonical functor*

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X,R) \to \mathrm{DM}_{\mathrm{h}}(X,R)$$

is an equivalence of categories.

Proof. We only sketch the proof. We see that it is sufficient to consider the cases where $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$, with p a prime (which is assumed odd if X is of unequal or positive characteristic). The case where $R = \mathbf{Q}$ is already known; see [CD12, Theorems 16.1.2 and 16.2.18]. The case of torsion coefficients follows from the fact that we may assume that p is prime to the residue characteristics of X (by Proposition A.3.4), and that we have a commutative diagram of the form

$$\begin{tikzcd}
& \mathrm{D}(X_{\text{ét}},\mathbf{Z}/p\mathbf{Z}) \arrow[dl] \arrow[dr] & \\
\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X,\mathbf{Z}/p\mathbf{Z}) \arrow[rr] & & \mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z}/p\mathbf{Z})
\end{tikzcd}$$

in which the non-horizontal functors are equivalences of categories (see [Ayo, Th. 4.1] and the preceding corollary, respectively). □

Proposition 5.4.8. *Let $f : X \to Y$ a morphism of finite type between quasi-excellent noetherian schemes of finite dimension. Then the functor*

$$\mathbf{R}f_* : \mathrm{DM}_h(X,R) \to \mathrm{DM}_\mathrm{h}(Y,R)$$

preserves small sums. In particular, this functor has a right adjoint. In the case where f is proper, we will denote by $f^!$ the right adjoint to $\mathbf{R}f_*$.

Proof. Using Proposition 5.3.5 and Corollary 5.3.11, we see that it is sufficient to prove the result in the case where $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$ for some prime p. For $R = \mathbf{Q}$ and any quasi-excellent scheme of finite dimension S, the triangulated categorie $\mathrm{DM}_\mathrm{h}(S,\mathbf{Q})$ is compactly generated and the functor $\mathbf{L}f^*$ preserves compact objects (see [CD12, Example 5.1.29(6) and Corollary 5.3.40]), which implies the claim. For $R = \mathbf{Z}/p\mathbf{Z}$, if p is invertible in the residue fields of Y, we conclude with Corollary 1.1.15 and Theorem 5.4.5. The general case follows from Proposition A.3.4. The existence of a right adjoint of $\mathbf{R}f_*$ is a direct consequence of the Brown representability theorem. □

Corollary 5.4.9. *Let $f : X \to Y$ a morphism of finite type between quasi-excellent noetherian schemes of finite dimension. For any object M of $\mathrm{DM}_\mathrm{h}(X,R)$ and any R-algebra R', there is a canonical isomorphism*

$$R' \otimes^{\mathbf{L}}_R \mathbf{R}f_*(M) \to \mathbf{R}f_*(R' \otimes^{\mathbf{L}}_R M).$$

Proof. Given a complex of R-modules C, we still denote by C the object of $\mathrm{DM}_\mathrm{h}(X,R)$ defined as the free Tate spectrum associated to the constant sheaf of complexes C. This defines a left Quillen functor from the projective model category on the category of complexes of R-modules (with quasi-isomorphisms ar weak equivalences, and degreewise surjective maps as fibrations) to the model category of Tate spectra. Therefore, we have a triangulated functor

$$\mathrm{D}(R\text{-Mod}) \to \mathrm{DM}_\mathrm{h}(S,R)\,, \quad C \mapsto C$$

which preserves small sums and is symmetric monoidal. By virtue of the preceding proposition, for any fixed M, we thus have a natural transformation between triangulated functors which preserve small sums:

$$C \otimes^{\mathbf{L}}_R \mathbf{R}f_*(M) \to \mathbf{R}f_*(C \otimes^{\mathbf{L}}_R M).$$

To prove that the map above is an isomorphism for any complex of R-modules C, as the derived category of R is compactly generated by R (seen as a complex concentrated in degree zero), it is sufficient to consider the case where $C = R$, which is trivial. □

Corollary 5.4.10. *Les X be a quasi-excellent noetherian scheme of finite dimension. Then, for any constructible motive M in $\mathrm{DM}_\mathrm{h}(X,R)$, the functor $\underline{\mathrm{Hom}}(M,-)$ preserves small sums. Furthermore, for any R-algebra R', we have canonical isomorphisms*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) \otimes^{\mathbf{L}}_R R' \simeq \mathbf{R}\underline{\mathrm{Hom}}_R(M,N \otimes^{\mathbf{L}}_R R')$$

for any object N in $\mathrm{DM}_\mathrm{h}(X,R)$.

Proof. It is sufficient to prove this in the case where M is of the form $M = \mathbf{L}f_\sharp(\mathbb{1}_Y)$ for a separated smooth morphism of finite type $f : Y \to X$. But then, we have

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) \simeq \mathbf{R}f_* f^*(N).$$

This corollary is thus a reformulation of Proposition 5.4.8 and Corollary 5.4.9. □

Corollary 5.4.11. *For any separated morphism of finite type $f : X \to Y$ between noetherian schemes of finite type, the functor*

$$f^! : \mathrm{DM_h}(Y,R) \to \mathrm{DM_h}(X,R)$$

preserves small sums, and, for any R-algebra R', there is a canonical isomorphism

$$f^!(M) \otimes^{\mathbf{L}}_R R' \simeq f^!(M \otimes^{\mathbf{L}}_R R').$$

Proof. For any constructible object C in $\mathrm{DM_h}(X,R)$, we have

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M)) \simeq \mathbf{R}\underline{\mathrm{Hom}}_R(f_!(C), M).$$

Using that the functor $f_!$ preserves constructible objects (see [CD12, Cor. 4.2.12]), we deduce from Proposition 5.4.8 and Corollary 5.4.10 the following computation, for any small family of objects M_i in $\mathrm{DM_h}(Y,R)$:

$$\begin{aligned}
\mathrm{Hom}(C, \bigoplus_i f^!(M_i)) &\simeq \mathrm{Hom}(\mathbb{1}_Y, \mathbf{R}\underline{\mathrm{Hom}}_R(C, \bigoplus_i f^!(M_i)))\\
&\simeq \mathrm{Hom}(\mathbb{1}_Y, \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R(C, \bigoplus_i f^!(M_i)))\\
&\simeq \mathrm{Hom}(\mathbb{1}_X, \mathbf{R}f_* \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M_i)))\\
&\simeq \mathrm{Hom}(\mathbb{1}_Y, \bigoplus_i \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M_i)))\\
&\simeq \mathrm{Hom}(\mathbb{1}_Y, \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R(f_!(C), M_i))\\
&\simeq \mathrm{Hom}(\mathbb{1}_Y, \mathbf{R}\underline{\mathrm{Hom}}_R(f_!(C), \bigoplus_i M_i))\\
&\simeq \mathrm{Hom}(f^!(C), \bigoplus_i M_i)\\
&\simeq \mathrm{Hom}(C, f^!(\bigoplus_i M_i)).
\end{aligned}$$

The change of coefficients formula is proved similarly (or with the same argument as in the proof of Corollary 5.4.9. □

5.5. h-motives and Grothendieck's 6 functors.

5.5.1. Let R be any commutative ring. Recall from [Voe96, Th. 4.2.5] that we get a canonical isomorphism in $\mathrm{DM}^{\mathit{eff}}_h(S,R)$:

$$\mathbb{1}_S(1) \simeq R \otimes^{\mathbf{L}} \mathbf{G}_m[-1]$$

where $\mathbf{G}_m$ is identified with the h-sheaf of abelian groups over S represented by the scheme $\mathbf{G}_m$.

This gives a canonical morphism of groups:

$$\begin{aligned}
c_1 : \mathrm{Pic}(S) = H^1_{\mathrm{Zar}}(S, \mathbf{G}_m) &\to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}_h(S,R)}(\mathbb{1}_S, \mathbb{1}_S(1)[2])\\
&\to \mathrm{Hom}_{\mathrm{DM}_h(S,R)}(\mathbb{1}_S, \mathbb{1}_S(1)[2])
\end{aligned}$$

so that the premotivic triangulated category $\mathrm{DM}_h(S,R)$ is oriented in the sense of Definition A.1.5.

Moreover, as a corollary of the results obtained above, we get:

Theorem 5.5.2. *The triangulated premotivic category* $\mathrm{DM}_h(-,R)$ *satisfies Grothendieck 6 functors formalism (Def. A.1.10) and the absolute purity property (Def. A.2.9).*

Proof. Taking into account Corollaries 5.4.9, 5.4.10 and 5.4.11, we see that we may assume $R=\mathbf{Z}$.

Consider the first assertion. Taking into account Theorem A.1.13, we have only to prove the localization property for $\mathrm{DM}_h(-,\mathbf{Z})$. Fix a closed immersion $i:Z\to S$. The analog of Proposition 2.3.4 for the h-topology obviously holds. This means we have to prove that for any smooth S-scheme X, the canonical map

$$\mathbf{Z}^h_S(X/X-X_Z)\to i_*\mathbf{Z}^h_Z(X_Z)$$

is an isomorphism in $\mathrm{DM}_h(-,\mathbf{Z})$. According to Proposition 5.3.12, together with 5.3.5 and 5.3.11, we are reduced to check this when $R=\mathbf{Q}$ or $R=\mathbf{Z}/p\mathbf{Z}$. In the first case, it follows from Theorem 5.4.2 and the localization property for DM_{B} – see [CD12]. In the second case, it follows from Theorem 5.4.5 and Theorem 4.3.1.

Concerning the second assertion, the absolute purity for $\mathrm{DM}_h(-,\mathbf{Z})$, we use the same argument as in the the proof of Theorem 4.6.1: using Theorem A.2.8, we can apply Proposition 5.3.12, together with 5.3.5 and 5.3.11 to reduced to the case where $R=\mathbf{Q}$ or $R=\mathbf{Z}/p\mathbf{Z}$. The first case follows from Theorem 5.4.2 and [CD12, Theorem 14.4.1] ; the second one follows from Theorem 5.4.5 and Theorem 4.6.1. □

5.6. **Transfers and traces.**

5.6.1. *Transfers*.– Consider the notations of Paragraph 5.4.3. Let X and Y be proper S-schemes and $\alpha\in c_S(X,Y)_\Lambda$ a finite S-correspondence. According to Proposition 5.4.4, we get a morphism of h-sheaves on $\mathscr{S}^{ft}_S$:

(5.6.1.a) $$\alpha_*:\underline{R}^h_S(X)\to\underline{R}^h_S(Y)$$

which induces a moorphism in $\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$:

$$\alpha_*:\Sigma^\infty\underline{R}^h_S(X)\to\Sigma^\infty\underline{R}^h_S(Y).$$

Let p and q be the respective structural morphisms of the S-schemes X and Y. Applying the functor $\underline{\mathrm{Hom}}(-,\mathbb{1}_S)$ to this map, we get a morphism in $\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$:

$$\alpha^*:q_*(\mathbb{1}_X)\to p_*(\mathbb{1}_Y).$$

Then we can apply to this functor the right adjoint ν^* of the adjunction (5.1.3.a) and, because it commutes with p_* and q_* and we have the isomorphism $\nu^*\mathbb{1}=\mathbb{1}$, the above morphism can be seen in $\mathrm{DM}_{\mathrm{h}}(S,R)$.

Given moreover any h-motive E over S, and using the projection formula – cf. Def. A.1.10, (2) and (5) – applied to the proper morphisms p and q, we obtain finally a canonical morphism:

$$q_*q^*(E)=q_*(\mathbb{1}_X)\otimes E\xrightarrow{\alpha^*\otimes Id_E}p_*(\mathbb{1}_Y)\otimes E=p_*p^*(E)$$

which is natural in E.

Definition 5.6.2. Consider the notations above. The following natural transformation of endofunctors of $\mathrm{DM}_{\mathrm{h}}(S,R)$

(5.6.2.a) $$\alpha^\star : q_*q^* \to p_*p^*$$

is called the cohomological h-transfer along the finite S-correspondence α.

The following results are easily derived from this definition:

Proposition 5.6.3. *Consider the above definition.*

(1) Normalisation.– *Consider a commutative diagram of schemes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

such that p and q are proper. Let α be the finite S-correspondence associated with the graph of f. Then the natural transformation $\alpha^\star$ is equal to the composite:

$$q_*q^* \xrightarrow{ad(f^*,f_*)} q_*f_*f^*q^* \simeq p_*p^*.$$

(2) Composition.– *For composable finite S-correspondences $\alpha \in c_S(X,Y)_\Lambda$, $\beta \in c_S(Y,Z)_\Lambda$ with X, Y, Z proper over S, one has: $\alpha^\star\beta^\star = (\beta\circ\alpha)^\star$.*

(3) Base change.– *Let $f : T \to S$ be a morphism of schemes, $\alpha \in c_S(X,Y)_\Lambda$ a finite S-correspondence between proper S-schemes and put $\alpha_T = f^*(\alpha)$ obtained using the premotivic structure on $\mathscr{S}^{cor}_\Lambda$. Let p (resp. q, p', q') be the structural morphism of X/S (resp. Y/S, $X\times_S T/T$, $Y\times_S T/T$), $f' = f\times_S T$. Then the following diagram commutes:*

$$\begin{tikzcd} f^*q_*q^* \arrow[rrr, "f^*.\alpha^\star"] \arrow[d, "Ex(f^*{,}q_*)"', "\sim"] & & & f^*p_*p^* \arrow[d, "\sim"', "Ex(f^*{,}p_*)"] \\ q'_*f'^*q^* \arrow[r, equal] & q'_*q'^* \arrow[r, "\alpha_T^\star"] & p'_*p'^* \arrow[r, equal] & p'_*p'^*q^* \end{tikzcd}$$

where the vertical maps are the proper base change isomorphisms – Def. A.1.10(4).

(4) Restriction.– *Let $\pi : S \to T$ be a proper morphism of schemes. Consider a finite S-correspondence $\alpha \in c_S(X,Y)_\Lambda$ between proper schemes and put $\alpha|_T = \pi_\sharp(\alpha)$ using the $\mathscr{S}^{ft}$-premotivic structure on $\mathscr{S}^{cor}_{\Lambda_S}$. Let p (resp. q) be the structural morphism of X/S (resp. Y/S), and put $p' = \pi\circ p$, $q' = \pi\circ q$. Then the following diagram is commutative:*

$$\begin{tikzcd} \pi_*q_*q^*\pi^* \arrow[r, "\pi_*.\alpha^\star.\pi^*"] \arrow[d, equal] & \pi_*p_*p^*\pi^* \arrow[d, equal] \\ q'_*q'^* \arrow[r, "(\alpha|_T)^\star"] & p'_*p'^* \end{tikzcd}$$

Proof. Property (1) and (2) are clear as they are obviously true for the morphism α_* of (5.6.1.a).

Similarly, property (3) (resp. (4)) follows from the fact the morphism (5.4.3.a) is compatible with the functor f^* (resp. the functor $\pi_\sharp$). This boils down to the fact that the graph functor[5] $\gamma : \mathscr{S}^{ft} \to \mathscr{S}^{cor}_{\Lambda}$ is a morphism of $\mathscr{S}^{ft}$-fibred category: see [CD12, 9.4.1]. □

5.6.4. Let $f : Y \to X$ be a morphism of schemes. Recall we say that f is Λ-universal if the fundamental cycle associated with Y is Λ-universal over X (Def. [CD12, 8.1.48]).

Let us denote by ${}^t f$ the cycle associated with the graph of f over X seen as a subscheme of $X \times_X Y$. Then, by the very definition, the following conditions are equivalent:

(i) f is finite Λ-universal;
(ii) the cycle ${}^t f$ is a finite X-correspondence from X to Y.

For matching the existing litterature, we introduce, the following definition, redundant with the previous one:

Definition 5.6.5. Let $f : Y \to X$ be a finite Λ-universal morphism of schemes. Using the preceding notations, we define the *trace of* f as the natural transformation of endofunctors of $\mathrm{DM_h}(X,R)$:

$$\mathrm{Tr}_f := ({}^t f)^\star : f_* f^* \to Id.$$

Remark 5.6.6. We will say that a morphism of schemes is pseudo-dominant if it sends any generic point to a generic point. Recall that a finite Λ-universal $f : Y \to X$ is in particular pseudo-dominant.

Let us recall the following example of finite Λ-universal morphisms of schemes:

(1) finite flat;
(2) finite pseudo-dominant morphisms whose aim is regular;
(3) finite pseudo-dominant morphisms whose aim is geometrically unibranch and has residue fields whose exponential characteristic is invertible in Λ.

5.6.7. One readily obtain from Proposition 5.6.3 that our trace maps are compatible with composition.

Recall that given a finite Λ-universal morphism $f : Y \to X$ and a generic point x of X, we can define an integer $\deg_x(f)$, the degree of f at x, by choosing any generic point y of Y such that $f(y) = x$ and putting:

$$\deg_x(f) := [\kappa(y) : \kappa(x)]$$

– see [CD12, 9.1.13]. We will say that f has *constant degree* d if for any generic point $x \in X$, $\deg_x(f) = d$.

Applying Proposition 5.6.3 to the particular case of traces, one gets the following formulas:

Proposition 5.6.8. *Consider the above definition.*

(1) Normalisation.– *Let $f : Y \to X$ be a finite étale morphism. Then the following diagram commutes:*

$$\begin{tikzcd} f_* f^* \arrow[r, "\mathrm{Tr}_f"] \arrow[d, "\alpha_f . \mathfrak{p}'_f"', "\sim"] & Id \\ f_! f^! \arrow[ur, "ad(f_!, f^!)"'] & \end{tikzcd}$$

where α_f and $\mathfrak{p}'_f$ are the isomorphisms from Definition A.1.10(2),(3).

[5]Recall: it is the identity on objects ans it associates to a morphism of separated S-schemes of finite type its S-graph seen as a finite S-correspondence.

(2) Composition.– *Let* $Z \xrightarrow{g} Y \xrightarrow{f} X$ *be finite* Λ*-universal morphisms. Then the following diagram commutes:*

$$\begin{tikzcd} f_*g_*g^*f^* \arrow[r, "f_*\mathrm{Tr}_g.f^*"] \arrow[d, equal] & f_*f^* \arrow[r, "\mathrm{Tr}_f"] & Id \arrow[d, equal] \\ (fg)_*(fg)^* \arrow[rr, "\mathrm{Tr}_{fg}"] & & Id. \end{tikzcd}$$

(3) Base change.– *Consider a pullback square of schemes:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

such that f is a finite flat morphism. Then, the following diagram is commutative:

$$\begin{tikzcd} \pi^*f_*f^* \arrow[rr, "\pi^*.\mathrm{Tr}_f"] \arrow[d, "Ex(\pi^*{,}p_*)"', "\sim"] & & \pi^* \arrow[d, equal] \\ f'_*\pi'^*f^* \arrow[r, equal] & f'_*f'^*\pi^* \arrow[r, "\mathrm{Tr}_{f'}.\pi^*"] & \pi^* \end{tikzcd}$$

where the left vertical map is the proper base change isomorphism.

(4) Degree formula.– *Let* $f : Y \to X$ *be a finite* Λ*-universal morphism of constant degree d, the following composite*

$$f_*f^* \xrightarrow{\mathrm{Tr}_f} Id \xrightarrow{ad(f^*,f_*)} f_*f^*$$

is equal to $d.Id$.

Proof. Point (1) follows from the fact that, in the category $\mathrm{Sh}_h(S,R)$, the sheaf $\underline{R}^h_X(Y)$ is strongly dualizable with itself as a dual and with duality pairings:

$$\underline{R}^h_X(Y) \otimes \underline{R}^h_X(Y) = \underline{R}^h_X(Y \times_X Y) \xrightarrow{({}^t\delta)_*} \underline{R}^h_X(Y) \xrightarrow{f_*} \underline{R}^h_X(X)$$
$$\underline{R}^h_X(X) \xrightarrow{({}^tf)_*} \underline{R}^h_X(Y) \xrightarrow{\delta_*} \underline{R}^h_X(Y \times_X Y) = \underline{R}^h_X(Y) \otimes \underline{R}^h_X(Y).$$

where δ is diagonal embedding (which is open and closed).

Point (2) is obtained from Proposition 5.6.3, properties (2) and (4). Point (3) is a special case of Proposition 5.6.3(3), given the fact that: $\pi^*({}^tf) = {}^tf'$ as f is flat – see [CD12], property (P3) of the tensor product of relative cycles in Paragraph 8.1.34. Point (4) follows from Proposition 5.6.3(1), (2) and the formula of Proposition 9.1.13 of [CD12]. $\square$

Remark 5.6.9. According to Corollary 5.4.6, this notion of trace generalizes the one introduced in [AGV73, XVII, sec. 6.2] in the case of finite morphisms, taking into account Remark 5.6.6.

Let us consider the more general case of a quasi-finite separated morphism $f : Y \to X$. According to the theorem of Nagata ([Con07]), there exists a factorization, $f = \bar{f} \circ j$, such that $\bar{f}$ is proper, thus finite according to Zariski's main theorem, and j is an open immersion.

We will say that f is *strongly Λ-universal* if there exists such a factorization such that in addition $\bar{f}$ is Λ-universal.[6]

In this condition, one checks easily using Proposition 5.6.8, properties (1) and (2), that the following composite is independent of the chosen factorization of f:

$$\mathrm{Tr}_f : f_! f^* = \bar{f}_! j_! j^* \bar{f}^* \xrightarrow{\bar{f}_! .ad(j_!, j^*).\bar{f}^*} f_! f^* = f_* f^* \xrightarrow{\mathrm{Tr}_{\bar{f}}} Id. \tag{5.6.9.a}$$

This composition is called the trace of f

Properties (1), (2), (3) of the preceding proposition immediately extend to this notion of trace.

However, this construction is not optimal as it is not clear that a flat quasi-finite separated morphism if strongly Λ-universal.

In particular, it only partially generalizes the construction of [AGV73, Th. 6.2.3] when $R = \mathbf{Z}/n\mathbf{Z}$ and X has residual characteristics prime to n. However, in the case where X is geometrically unibranch, and has residual characteristics prime to n, any quasi-finite separated pseudo-dominant morphism is strongly Λ-universal (cf Rem. 5.6.6). Thus, in this case, our notion does generalize the finer notion of trace introduced in [AGV73, 6.2.5, 6.2.6].

5.7. **Local localisations.**

5.7.1. In the followings, we give some complements on localization of abstract triangulated categories. We fix a commutative ring A and a multiplicative system $S \subset A$.

For a triangulated category T, we shall denote by $T^\sharp$ its idempotent completion (with its canonical triangulated structure; see [BS01]).

Proposition 5.7.2. *Let T be a triangulated category and $S \subset T$ a thick subcategory of T. Then $U^\sharp$ is a thick subcategory of $T^\sharp$ and the natural triangulated functor*

$$(T/U)^\sharp \to \left(T^\sharp/U^\sharp\right)^\sharp$$

is an equivalence of categories.

Proof. Both functors $T \to (T/U)^\sharp$ and $T \to \left(T^\sharp/U^\sharp\right)^\sharp$ share the same universal property, namely of being the universal functor from T to an idempotent complete triangulated category in which any object of U becomes null. □

Corollary 5.7.3. *Given a triangulated category T and a thick subcategory U of T, an object of T belongs to U if and only if its image is isomorphic to zero in the triangulated category $\left(T^\sharp/U^\sharp\right)^\sharp$.*

Proof. As U is thick in T, an object of T is in U if and only if its image in the Verdier quotient T/U is trivial. On the other hand, the preceding proposition implies in particular that the natural functor

$$T/U \to \left(T^\sharp/U^\sharp\right)^\sharp$$

is fully faithful, which implies the assertion. □

5.7.4. Let T be an A-linear triangulated category. We define a new triangulated category $T \otimes_A S^{-1}A$ as follows. The objects of $T \otimes_A S^{-1}A$ are those of T, and morphisms from X to Y are given by the formula

$$\mathrm{Hom}_{T \otimes_A S^{-1}A}(X,Y) = \mathrm{Hom}_T(X,Y) \otimes_A S^{-1}A$$

[6] This implies in particular that f is Λ-universal according to [CD12, Cor. 8.2.6]. The converse is not true.

(with the obvious composition law. We have an obvious triangulated functor

$$T \to T \otimes_A S^{-1}A \tag{5.7.4.a}$$

which is the identity on objects and which is defined by the canonical maps

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}_T(X,Y) \otimes_A S^{-1}A$$

on arrows. The distinguished triangles of $T \otimes_A S^{-1}A$ are the triangles which are isomorphic to some image of a distinguished triangle of T by the functor (5.7.4.a).

Given an object X of T and an element $f \in S$, we write $f : X \to X$ for the map $f.1_X$, and we shall write X/f for some choice of its cone. We write $T_{S\text{-}tors}$ for the smallest thick subcategory of T which contains the cones of the form X/f for any object X and any f in S, the objects of which will be called *S-torsion objects of T*. The functor (5.7.4.a) clearly sends S-torsion objects to zero, and thus induces a canonical triangulated functor

$$T/T_{S\text{-}tors} \to T \otimes_A S^{-1}A\,. \tag{5.7.4.b}$$

Proposition 5.7.5. *The functor* (5.7.4.b) *is an equivalence of categories.*

Proof. One readily checks that T is $S^{-1}A$-linear if and only if $T_{S\text{-}tors} \simeq 0$. Therefore, both functors $T \to T/T_{S\text{-}tors}$ and (5.7.4.a) share the same universal property: these are the universal A-linear triangulated functors from T to an $S^{-1}A$-linear triangulated category. □

Corollary 5.7.6. *We have a canonical equivalence of A-linear triangulated categories*

$$(T \otimes_A S^{-1}A)^\sharp \simeq (T^\sharp \otimes_A S^{-1}A)^\sharp\,.$$

Proof. This follows again from the fact that, by virtue of Propositions 5.7.2 and 5.7.5, these two categories are the universal A-linear idempotent complete triangulated categories under T in which the S-torsion objects are trivial. □

Proposition 5.7.7. *Let T be an A-linear triangulated category and U a thick subcategory of T. Given a prime ideal $\mathfrak{p}$ in A, we write $T_\mathfrak{p} = T \otimes_A A_\mathfrak{p}$. For an object X of T, the following conditions are equivalent.*

(i) *The object X belongs to U.*
(ii) *For any maximal ideal $\mathfrak{m}$ in A, the image of X in $(T/U)_\mathfrak{m}$ is trivial.*
(iii) *For any maximal ideal $\mathfrak{m}$ of A, the image of X in $(T^\sharp_\mathfrak{m}/U^\sharp_\mathfrak{m})^\sharp$ is trivial.*

Proof. The equivalence between conditions (ii) and (iii) readily follows from Corollaries 5.7.3 and 5.7.6. The equivalence between conditions (i) and (ii) comes from the fact that the localisations $A_\mathfrak{m}$ form a covering for the flat topology and from the Yoneda lemma. □

5.7.8. Let S be a noetherian scheme. For any prime ideal $\mathfrak{p}$ of $\mathbf{Z}$, we have a fully faithful functor

$$\big(\mathrm{DM}_{\mathrm{h},c}(S,\mathbf{Z})_\mathfrak{p}\big)^\sharp \to \big(\mathrm{DM}_\mathrm{h}(S,\mathbf{Z})_\mathfrak{p}\big)^\sharp \tag{5.7.8.a}$$

Definition 5.7.9. An object M of $\mathrm{DM}_\mathrm{h}(S,\mathbf{Z})$ will be called $\mathfrak{p}$-*constructible* if its image in $\big(\mathrm{DM}_\mathrm{h}(S,\mathbf{Z})_\mathfrak{p}\big)^\sharp$ lies in the essential image of the functor (5.7.8.a).

Let us state explicitly the corollary that we will use below:

Corollary 5.7.10. *Let S be a noetherian scheme and M be an object of* $\mathrm{DM}_\mathrm{h}(S,\mathbf{Z})$. *Then the following conditions are equivalent:*

(i) *M is constructible;*
(ii) *for any maximal ideal $\mathfrak{p} \in \mathrm{Spec}(\mathbf{Z})$, M is $\mathfrak{p}$-constructible.*

Proof. We just apply the preceding proposition to the $\mathbf{Z}$-linear category $T = \mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ and its thick subcategory $U = \mathrm{DM}_{\mathrm{h},c}(S,\mathbf{Z})$. □

Proposition 5.7.11. *Let p be a prime number and X a noetherian scheme of characteristic p. An object M of $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ is (p)-constructible if and only if it is (0)-constructible.*

Proof. The Artin-Schreier short exact sequence (see the proof of Proposition A.3.1) implies that the category $\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ is $\mathbf{Z}[1/p]$-linear, so that we have

$$\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})_{(p)} = \mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z}) \otimes \mathbf{Q},$$

and similarly for $\mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z})$. □

Remark 5.7.12. When $\mathfrak{p} = (0)$, the functor $\rho_{\mathfrak{p}}^*$ which appears in this corollary coincide with the functor ρ^* of Corollary 5.3.9 in the case $R' = R \otimes_{\mathbf{Z}} \mathbf{Q}$.

5.8. **Constructible h-motives.** In this subsection, we will simplify the notations by dopping the symbols $\mathbf{L}$ and $\mathbf{R}$; in other words, by default, all the functors will be the derived ones. We will prove the main theorems about constructible h-motives: their stability by the 6 operations (Th. 5.8.8 and its corollary) and the duality theorem (Th. 5.8.12). The stability statement boils down to the stability with respect to direct image. This result, based on an argument of Gabber, is intricate and we divide its proof with the help of the following two results. The first one can be found either in [Ayo07, Lem. 2.2.23] or in [CD12, Prop. 4.2.13]:

Proposition 5.8.1. *Let X be a noetherian scheme. The category $\mathrm{DM}_{\mathrm{h},c}(X,R)$ is the smallest thick triangulated subcategory of $\mathrm{DM}_{\mathrm{h}}(X,R)$ which contains the objects of the form $f_*(R_{X'}(n))$ where $f : X' \to X$ is a projective morphism and $n \in \mathbf{Z}$.*

The second result used in the proof of the forthcoming theorem 5.8.8 is an elaboration of an argument of Gabber used in the étale torsion case (see [ILO12, XIII, section 3]).

Lemma 5.8.2 (Gabber's Lemma)**.** *Let X be a quasi-excellent scheme, and $\mathfrak{p}$ a prime ideal of $\mathbf{Z}$. Assume that, for any point x of X, the exponent characteristic of the residue field $\kappa(x)$ is not in $\mathfrak{p}$. Then, for any dense open immersion $j : U \to X$, the h-motive $j_*(\mathbb{1}_U)$ is $\mathfrak{p}$-constructible.*

Proof. We will use the following geometrical consequence of the *local uniformisation theorem prime to* $\mathfrak{p}$ of Gabber (see [ILO12, VII, 1.1 and IX, 1.1]):

Lemma 5.8.3. *Let $j : U \to X$ be a dense open immersion such that X is quasi-excellent, and $\mathfrak{p}$ a prime ideal of $\mathbf{Z}$. Assume that, for any point x of X, the exponent characteristic of the residue field $\kappa(x)$ is not in $\mathfrak{p}$. Then, there exists the following data:*

(i) *a finite h-cover $\{f_i : Y_i \to X\}_{i \in I}$ such that for all i in I, f_i is a morphism of finite type, the scheme Y_i is regular, and $f_i^{-1}(U)$ is either Y_i itself or the complement of a strict normal crossing divisor in Y_i; we shall write*

$$f : Y = \coprod_{i \in I} Y_i \to X$$

for the induced global h-cover;

(ii) *a commutative diagram*

$$\begin{tikzcd} X''' \arrow[rr, "g"] \arrow[d, "q"'] & & Y \arrow[d, "f"] \\ X'' \arrow[r, "u"] & X' \arrow[r, "p"] & X \end{tikzcd} \tag{5.8.3.a}$$

in which: p is a proper birational morphism, u is a Nisnevich cover, and q is a flat finite surjective morphism of degree not in $\mathfrak{p}$.

Let T (resp. T') be a closed subscheme of X (resp. X') and assume that for any irreducible component T_0 of T, the following inequality is satisfied:

$$\operatorname{codim}_{X'}(T') \geq \operatorname{codim}_X(T_0),$$

Then, possibly after shrinking X in an open neighbourhood of the generic points of T in X, one can replace X'' by an open cover and X''' by its pullback along this cover, in such a way that we have in addition the following properties:

(iii) *$p(T') \subset T$ and the induced map $T' \to T$ is finite and sends any generic point to a generic point;*

(iv) *if we write $T'' = u^{-1}(T')$, the induced map $T'' \to T'$ is an isomorphism.*

Points (i) and (ii) are proved in [ILO12, Par. 3.2.1]. Then points (iii) and (iv) are proved in [CD12, proof of Lem. 4.2.14].

5.8.4. We introduce the following notations: for any scheme Y, we let $\mathscr{T}_0(Y)$ be the subcategory of $\mathrm{DM}_{\mathrm{h}}(Y, \mathbf{Z})$ made of $\mathfrak{p}$-constructible objects K. Then $\mathscr{T}_0$ becomes a fibred subcategory of $\mathrm{DM}_{\mathrm{h}}(-, \mathbf{Z})$ and we can moreover check the following properties:

(a) for any scheme Y in Sch, $\mathscr{T}_0(Y)$ is a triangulated thick subcategory of $\mathrm{DM}_{\mathrm{h}}(Y, \mathbf{Z})$ which contains the objects of the form $\mathbb{1}_Y(n)$, $n \in \mathbf{Z}$;

(b) for any separated morphism of finite type $f : Y' \to Y$ in Sch, $\mathscr{T}_0$ is stable under $f_!$;

(c) for any dense open immersion $j : V \to Y$, with Y regular, which is the complement of a strict normal crossing divisor, $j_*(\mathbb{1}_V)$ is in $\mathscr{T}_0(V)$.

Indeed: (a) is obvious, (b) follows from the fact the functor $f_!$ perserves constructible motives, while (c) comes from the absolute purity property for $\mathrm{DM}_{\mathrm{h}}(-, \mathbf{Z})$; see Theorem 5.5.2. With this notation, we have to prove that $j_*(\mathbb{1}_U)$ is in $\mathscr{T}_0$.

Following the argument of [ILO12, 3.1.3], it is sufficient to prove by induction on $c \geq 0$ that here exists a closed subscheme $T \subset X$ of codimension $> c$ such that the restriction of $j_*(\mathbb{1}_U)$ to $(X - T)$ is in $\mathscr{T}_0$.

Indeed, if this is the case, let us chose a closed subset T_c of X satisfying the condition above with respect to an arbitrary integer $c \geq 0$. As X is noetherian, we get that X is covered by the family of open subschemes $(X - T_c)$ indexed by $c \geq 0$. Moreover, X is quasi-compact so that only a finite number of these open subschemes are sufficient to cover X. Thus we can conclude that $j_*(\mathbb{1}_U)$ is in $\mathscr{T}_0$ iteratively using the Mayer-Vietoris exact triangle and property (a) of 5.8.4.

The case where $c = 0$ is clear: we can choose T such that $(X - T) = U$. If $c > 0$, we choose a closed subscheme T of X, of codimension $> c - 1$, such that the restriction of $j_*(\mathbb{1}_U)$ to $(X - T)$ is in $\mathscr{T}_0$. It is then sufficient to find a dense open subscheme V of X, which contains all the generic points of T, and such that the restriction of $j_*(\mathbb{1}_U)$ to V is in $\mathscr{T}_0$: for such a V, we shall obtain that the restriction of $j_*(\mathbb{1}_U)$ to $V \cup (X - T)$ is in $\mathscr{T}_0$, the complement of $V \cup (X - T)$ being the support of a closed subscheme of codimension $> c$ in X. In particular, using the smooth base change isomorphism (for open immersions), we can always replace X by a generic neighbourhood of T. It is sufficient to prove that, possibly after shrinking X as above, the pullback of $j_*(\mathbb{1}_U)$ along $T \to X$ is in $\mathscr{T}_0$ (as we already know that its restriction to $(X-T)$ is in $\mathscr{T}_0$).

We may assume that T is purely of codimension c. We may assume that we have data as in points (i) and (ii) of Lemma 5.8.3. We let $j' : U' \to X'$ denote the pullback of j along $p : X' \to X$. Then, we can find, by induction on c, a closed subscheme T' in X', of codimension $> c-1$, such that the restriction of $j'_*(\mathbb{1}_{U'})$ to $(X'-T')$ is in $\mathscr{T}_0$. By shrinking X, we may assume that conditions (iii) and (iv) of Lemma 5.8.3 are fulfilled as well.

Given any morphism $i : Z \to W$ of X-schemes, we consider the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "\pi"'] & W \arrow[d] & W_U \arrow[l, "j_W"'] \arrow[d] \\ & X & U, \arrow[l, "j"'] \end{tikzcd}$$

where the right hand square is cartesian, and we define the following h-motive of $\mathrm{DM_h}(X,R)$:

$$\varphi(W,Z) := \pi_*\, i^*\, j_{W,*}(\mathbb{1}_{W_U}).$$

This notation is slightly abusive but it will most of the time be used when i is the immersion of a closed subscheme. This construction is contravariantly functorial: given any commutative diagram of X-schemes:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ W' \arrow[r] & W \end{tikzcd}$$

we get a natural map $\varphi(W,Z) \to \varphi(W',Z')$. Remember that we want to prove that $\varphi(X,T)$ is in $\mathscr{T}_0$. This will be done via the following lemmas (which hold assuming all the conditions stated in Lemma 5.8.3 as well as our inductive assumptions).

Lemma 5.8.5. *The cone of the map $\varphi(X,T) \to \varphi(X',T')$ is in $\mathscr{T}_0$.*

The map $\varphi(X,T) \to \varphi(X',T')$ factors as

$$\varphi(X,T) \to \varphi(X',p^{-1}(T)) \to \varphi(X',T').$$

By the octahedral axiom, it is sufficient to prove that each of these two maps has a cone in $\mathscr{T}_0$.

We shall prove first that the cone of the map $\varphi(X',p^{-1}(T)) \to \varphi(X',T')$ is in $\mathscr{T}_0$. Given an immersion $a : S \to X'$, we shall write

$$M_S = a_!\, a^*(M).$$

We then have distinguished triangles

$$M_{p^{-1}(T)-T'} \to M_{p^{-1}(T)} \to M_{T'} \to M_{p^{-1}(T)-T'}[1].$$

For $M = j'_*(\mathbb{1}_{U'})$ (recall j' is the pullback of j along p) the image of this triangle by p_* gives a distinguished triangle

$$p_*(M_{p^{-1}(T)-T'}) \to \varphi(X',p^{-1}(T)) \to \varphi(X',T') \to p_*(M_{p^{-1}(T)-T'})[1].$$

As the restriction of $M = j'_*(\mathbb{1}_{U'})$ to $X'-T'$ is in $\mathscr{T}_0$ by assumption on T', the object $M_{p^{-1}(T)-T'}$ is in $\mathscr{T}_0$ as well (by property (b) of 5.8.4), from which we deduce that

$p_*(M_{p^{-1}(T)-T'})$ is in $\mathscr{T}_0$ (using the condition (iii) of Lemma 5.8.3 and property (b) of 5.8.4).

Let V be a dense open subscheme of X such that $p^{-1}(V) \to V$ is an isomorphism. We may assume that $V \subset U$, and write $i : Z \to U$ for the complement closed immersion. Let $p_U : U' = p^{-1}(U) \to U$ be the pullback of p along j, and let $\bar{Z}$ be the reduced closure of Z in X. We thus get the commutative squares of immersions below,

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "i"'] & \bar{Z} \arrow[d, "l"] \\ U \arrow[r, "j"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z' \arrow[r, "k'"] \arrow[d, "i'"'] & \bar{Z}' \arrow[d, "l'"] \\ U' \arrow[r, "j'"'] & X' \end{tikzcd}$$

where the square on the right is obtained from the one on the left by pulling back along $p : X' \to X$. Recall that the triangulated motivic category $\mathrm{DM}_{\mathrm{h}}(-,\mathbf{Z})$ satisfies cdh-descent (see [CD12, Prop. 3.3.10]). Thus, as p is an isomorphism over V, we get the homotopy cartesian square below.

$$\begin{tikzcd} \mathbb{1}_U \arrow[r] \arrow[d] & p_{U,*}(\mathbb{1}_{U'}) \arrow[d] \\ i_* i^*(\mathbb{1}_Z) \arrow[r] & i_* i^* p_{U,*}(\mathbb{1}_{U'}) \end{tikzcd}$$

If $a : T \to X$ denotes the inclusion, applying the functor $a_* a^* j_*$ to the commutative square above, we see from the proper base change formula and from the identification $j_* i_* \simeq l_* k_*$ that we get a commutative square isomorphic to the following one

$$\begin{tikzcd} \varphi(X,T) \arrow[r] \arrow[d] & \varphi(X', p^{-1}(T)) \arrow[d] \\ \varphi(\bar{Z}, \bar{Z} \cap T) \arrow[r] & \varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T)), \end{tikzcd}$$

which is thus homotopy cartesian as well. It is sufficient to prove that the two objects $\varphi(\bar{Z}, \bar{Z} \cap T)$ and $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ are in $\mathscr{T}_0$. It follows from the proper base change formula that the object $\varphi(\bar{Z}, \bar{Z} \cap T)$ is canonically isomorphic to the restriction to T of $l_* k_*(\mathbb{1}_Z)$. As $\dim \bar{Z} < \dim X$, we know that the object $k_*(\mathbb{1}_Z)$ is in $\mathscr{T}_0$. By property (b) of 5.8.4, we obtain that $\varphi(\bar{Z}, \bar{Z} \cap T)$ is in $\mathscr{T}_0$. Similarly, the object $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is canonically isomorphic to the restriction of $p_* l'_* k'_*(\mathbb{1}_{Z'})$ to T, and, as $\dim \bar{Z}' < \dim X'$ (because, p being an isomorphism over the dense open subscheme V of X, $\bar{Z}'$ does not contain any generic point of X'), $k'_*(\mathbb{1}_{Z'})$ is in $\mathscr{T}_0$. We deduce again from property (b) of 5.8.4 that $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is in $\mathscr{T}_0$ as well, which achieves the proof of the lemma.

Lemma 5.8.6. *The map $\varphi(X',T') \to \varphi(X'',T'')$ is an isomorphism in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$.*

Condition (iv) of Lemma 5.8.3 can be reformulated by saying that we have the Nisnevich distinguished square below.

$$\begin{tikzcd} X'' - T'' \arrow[r] \arrow[d] & X'' \arrow[d, "v"] \\ X' - T' \arrow[r] & X' \end{tikzcd}$$

This lemma follows then by Nisnevich excision ([CD12, 3.3.4]) and smooth base change (for étale maps).

In the next lemma, we call $\mathfrak{p}$-*quasi-section* of a morphism $f : K \to L$ in $\mathrm{DM_h}(X,\mathbf{Z})$ any morphism $s : L \to K$ such that there exists an integer n, not in $\mathfrak{p}$, and such that: $f \circ s = n.Id$.

Lemma 5.8.7. *Let T''' be the pullback of T'' along the finite surjective morphism $X''' \to X''$. The map $\varphi(X'',T'') \to \varphi(X''',T''')$ admits a $\mathfrak{p}$-quasi-section.*

We have the following pullback squares

$$\begin{tikzcd} T''' \arrow[r, "t"] \arrow[d, "r"'] & X''' \arrow[d, "q"] & U''' \arrow[l, "j'''"'] \arrow[d, "q_U"] \\ T'' \arrow[r, "s"] & X'' & U' \arrow[l, "j''"'] \end{tikzcd}$$

in which j'' and j''' denote the pullback of j along pu and puq respectively, while s and t are the inclusions. By the proper base change formula applied to the left hand square, we see that the map $\varphi(X'',T'') \to \varphi(X''',T''')$ is isomorphic to the image of the map

$$j''_*(\mathbb{1}_{U''}) \to q_* q^* j''_*(\mathbb{1}_{U''}) \to q_* j'''_*(\mathbb{1}_{U'''}).$$

by $f_* s^*$, where $f : T'' \to T$ is the map induced by p (note that f is proper as $T'' \simeq T'$ by assumption). As $q_* j'''_* \simeq j''_* q_{U,*}$, we are thus reduced to prove that the unit map

$$\mathbb{1}_{U''} \to q_{U,*}(\mathbb{1}_{U'''})$$

admits a p-quasi-section. By property (iii) of Lemma 5.8.3, q_U is a flat finite surjective morphism of degree n not in $\mathfrak{p}$. Thus the $\mathfrak{p}$-quasi-section is given by the trace map (Definition 5.6.5) associated with q_U, taking into account the *degree formula* of Proposition 5.6.8.

Now, we can finish the proof of Theorem 5.8.2. Let us apply the functoriality of the construction φ with respect to the following commutative squares:

$$\begin{tikzcd} T''' \arrow[r, equal] \arrow[d, "t"'] & T''' \arrow[r] \arrow[d, "a"'] & T \arrow[d] \\ X''' \arrow[r, "g"] & Y \arrow[r, "f"] & X \end{tikzcd}$$

where $T''' = q^{-1}u^{-1}(T')$, t is the natural map and $a = g \circ t$, we get the following commutative diagram of $\mathrm{DM_h}(X,\mathbf{Z})$:

$$\begin{tikzcd} \varphi(X,T) \arrow[rr, "(1)"] \arrow[dr] & & \varphi(X''',T''') \\ & \varphi(Y,T''') \arrow[ur] & \end{tikzcd}$$

We consider the image of that diagram through the functor

$$\bar{\rho} : \mathrm{DM_h}(X,\mathbf{Z}) \to \mathrm{DM_h}(X,\mathbf{Z})/\mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z}) \to \left(\mathrm{DM_h}(X,\mathbf{Z})/\mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z})\right)_{\mathfrak{p}}.$$

By virtue of Proposition 5.7.7, we have to show that the image of $\varphi(X,T)$ under $\bar{\rho}$ is 0. According to lemmas 5.8.5, 5.8.7, and 5.8.6, the image of (1) under $\bar{\rho}$ is a split monomorphism. Thus it is sufficient to prove that this image is the zero map, and according to the commutativity of the above diagram, this will follow if we prove that $\bar{\rho}(\varphi(Y,T''')) = 0$, which amounts to prove that $\varphi(Y,T''')$ is $\mathfrak{p}$-constructible.

We come back to the definition of $\varphi(Y, T''')$: considering the following commutative diagram,

$$\begin{tikzcd} T''' \arrow[r, "a"] \arrow[dr, "\pi"'] & Y \arrow[d, "f"] & Y_U \arrow[l, "j_Y"'] \arrow[d] \\ & X & U, \arrow[l, "j"'] \end{tikzcd}$$

we have: $\varphi(Y, T''') = \pi_* a^* j_{Y,*}(\mathbb{1}_{Y_U})$. By assumption, the morphism π is finite – this follows more precisely from the following conditions of Lemma 5.8.3: (ii) saying that q is finite, (iii) and (iv). Thus by assumption on j_Y (see point (i) of Lemma 5.8.3), we obtain that $\varphi(Y, T''')$ is $\mathfrak{p}$-constructible, according to properties (b) and (c) stated in Paragraph 5.8.4. This achieves the proof of Gabber's Lemma. □

Theorem 5.8.8. *Let $f : Y \to X$ be a morphism of finite type such that X is a quasi-excellent scheme. Then for any constructible* h*-motive K of* $\mathrm{DM}_{\mathrm{h}}(Y, R)$*, $f_*(K)$ is constructible in* $\mathrm{DM}_{\mathrm{h}}(X, R)$*.*

Proof. The case where f is proper is already known from [CD12, Prop. 4.2.11]. Then, a well-known argument allows to reduce to prove that for any dense open immersion $j : U \to X$, the h-motive $j_*(R_U)$ is constructible. Indeed, assume this is known. We want to prove that $f_*(K)$ is constructible whenever K is constructible. According to Proposition 5.8.1, and because f_* commutes with Tate twists, it is sufficient to consider the case $K = R_Y$. Moreover, we easily conclude from Corollary 5.4.9 that we may assume that $R = \mathbf{Z}$. Then, as this property is assumed to be known for dense open immersions, by an easy Mayer-Vietoris argument, we see that the condition that $f_*(R_Y)$ is constructible is local on Y and X with respect to the Zariski topology. Therefore, we may assume that X and Y are affine, thus f is affine ([GD61, (1.6.2)]) and in particular quasi-projective ([GD61, (5.3.4)]): it can be factored as $f = \bar{f} \circ j$ where f is projective and j is a dense open immersion. The case of $\bar{f}$ being already known from [CD12, Prop. 4.2.11], we may assume $f = j$.

Thus, as j_* commutes with Tate twist, it is sufficient to prove that for any dense open immersion $j : U \to X$, with X a quasi-excellent, the h-motive $j_*(R_U)$ is constructible. Applying Corollary 5.7.10, it is sufficient to prove that, given any prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathbf{Z})$, the h-motive $j_*(\mathbb{1}_U)$ is $\mathfrak{p}$-constructible.

The case where $\mathfrak{p} = (0)$ directly follows from Gabber's Lemma 5.8.2. Assume now that $\mathfrak{p} = (p)$ for a prime number $p > 0$. Let us consider the following cartesian square of schemes, in which $X_p = X \times \mathrm{Spec}(\mathbf{Z}[1/p])$:

$$\begin{tikzcd} U_p \arrow[r, "i_U"] \arrow[d, "j_p"'] & U \arrow[d, "j"] & U' \arrow[l, "j_U"'] \arrow[d, "j'"] \\ X_p \arrow[r, "i_X"] & X & X' \arrow[l, "j_X"'] \end{tikzcd}$$

Then we can consider the following localization distinguished triangle:

$$j_{X!} j_X^* j_*(\mathbb{1}_U) \to j_*(\mathbb{1}_U) \to i_{X*} i_X^* j_*(\mathbb{1}_U) \xrightarrow{+1}$$

so that it is sufficient to prove that the first and third motives in the above triangle are $\mathfrak{p}$-constructible. Note that the functors $j_{X!}$ and i_{X*} preserve, $\mathfrak{p}$-constructible objects, so that it is sufficient to prove that $i_X^* j_*(\mathbb{1}_U)$ and $j_X^* j_*(R_U)$ are $\mathfrak{p}$-constructible.

The object $i_X^* j_*(\mathbb{1}_U)$ being (0)-constructible, it is $\mathfrak{p}$-constructible, by virtue of Proposition 5.7.11. It remains to prove that the following h-motive is $\mathfrak{p}$-constructible:

$$j_X^* j_*(R_U) = j'_*(R_{U'})$$

(for the isomorphism, we have used the smooth base change theorem, which is trivially true in $\mathrm{DM_h}$, by construction). Thus, we are finally reduced to Gabber's Lemma 5.8.2, and this concludes. □

Corollary 5.8.9. *The six operations preserve constructibility in* $\mathrm{DM_h}(-,R)$ *over quasi-excellent noetherian schemes of finite dimension. In other words, we have the following stability properties.*

(a) *For any quasi-excellent noetherian scheme of finite dimension X, any constructible objects M and N in $\mathrm{DM_h}(X,R)$, the objects $M \otimes_R N$ and $\underline{\mathrm{Hom}}_R(M,N)$ are constructible.*
(b) *For any separated morphism of finite type between quasi-excellent noetherian schemes of finite dimension $f : X \to Y$, and for any constructible object M of $\mathrm{DM_h}(X,R)$, the objects $f_*(M)$ and $f_!(M)$ are constructible, and for any constructible object N of $\mathrm{DM_h}(Y,R)$, the objects $f^*(N)$ and $f^!(N)$ are constructible.*

Proof. The fact that f^* preserves constructibility is obvious. The case of f_* follows from the preceding theorem. The tensor product also preserves constructibility on the nose. To prove that $\underline{\mathrm{Hom}}_R(M,N)$ is constructible for any constructible objects M and N in $\mathrm{DM_h}(X,R)$, we may assume that $M = f_\sharp(\mathbb{1}_Y)$ for a separated smooth morphism of finite type $f : Y \to X$. In this case, we have the isomorphism

$$\underline{\mathrm{Hom}}_R(M,N) \simeq f_* f^*(N),$$

from which we get the expected property. The fact that the functors of the form $f_!$ preserve constructibility is well known (see for instance [CD12, Cor. 4.2.12]). Let $f : X \to Y$ be a separated morphism of finite type between quasi-excellent noetherian schemes of finite dimension. The property that $f^!$ preserves constructibility is local on X and on Y with respect to the Zariski topology (see [CD12, Lemma 4.2.27]), so that we may assume that f is affine. From there, we see that we may assume that f is an open immersion, or that f is the projection of the projective space $\mathbf{P}^n_Y$ to the base, or that f is a closed immersion. The case of an open immersion is trivial. In the case where f is a projective space of dimension n, the purity isomorphism $f^! \simeq f^*(n)[2n]$ allows to conclude. Finally, if $f = i$ is a closed immersion with open complement $j : U \to Y$, then we have distinguished triangles

$$i_* i^!(M) \to M \to j_* j^*(M) \to$$

from which deduce that $i_* i^!(M)$ is constructible, and thus that $i^!(M) \simeq i^* i_* i^!(M)$ is constructible, whenever M has this property. □

5.8.10. An object U of $\mathrm{DM_h}(X,R)$ will be said to be *dualizing* if it has the following two properties:

(i) U is constructible;
(ii) For any constructible object M in $\mathrm{DM_h}(X,R)$, the canonical morphism

$$M \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M,U),U)$$

is an isomorphism.

Lemma 5.8.11. *Let X be a quasi-excellent noetherian scheme of finite dimension.*

(i) *If an object U of $\mathrm{DM_h}(X,\mathbf{Z})$ is dualizing, then, for any commutaitve ring R, the (derived) tensor product $R\otimes U$ is dualizing in $\mathrm{DM_h}(X,R)$.*

(ii) *A constructible object U of $\mathrm{DM_h}(X,R)$ is dualizing if an only if $\mathbf{Q}\otimes U$ is dualizing in $\mathrm{DM_h}(X,\mathbf{Q})$ and, for any prime p, U/p is dualizing in $\mathrm{DM_h}(X,\mathbf{Z}/p\mathbf{Z})$.*

Proof. Assume that the object U of $\mathrm{DM_h}(X,\mathbf{Z})$ is dualizing. To prove that the canonical map

$$M \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M,R\otimes U),R\otimes U)$$

is invertible for any constructible object M in $\mathrm{DM_h}(X,R)$, we may assume that

$$M = f_\sharp(R_Y)\simeq R\otimes f_\sharp(\mathbf{Z}_Y)$$

for a separated smooth morphism of finite type $f: Y\to X$. In particular, we may assume that $M=R\otimes C$ for a constructible object C in $\mathrm{DM_h}(X,\mathbf{Z})$. But then, by virtue of Corollary 5.4.10, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C,U),U)\otimes R\simeq \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M,R\otimes U),R\otimes U)\,,$$

from which we conclude that $R\otimes U$ is dualizing. The proof of the second assertion is similar. Indeed, for any constructible object C of $\mathrm{DM_h}(X,\mathbf{Z})$, by virtue of Corollary 5.3.11, we have canonical isomorphisms

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C,U),U)\otimes\mathbf{Q}\simeq \underline{\mathrm{Hom}}_{\mathbf{Q}}(\underline{\mathrm{Hom}}_{\mathbf{Q}}(\mathbf{Q}\otimes C,\mathbf{Q}\otimes U),\mathbf{Q}\otimes U)\,,$$

and, by Proposition 5.3.5, for any positive integer n, canonical isomorphisms

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C,U),U)/n\simeq \underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(\underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(C/n,U/n),U/n)\,.$$

By virtue of Proposition 5.3.12, this readily implies assertion (*ii*). □

Theorem 5.8.12. *Let B be an excellent noetherian scheme of dimension ≤ 2 (or, more generally, which admits wide resolution of singularities up to quotient singularities in the sense of* [CD12, Def. 4.1.9]*).*

(a) *For any regular B-scheme of finite type S, an object U of $\mathrm{DM_h}(S,R)$ is dualizing if and only if it is constructible and $\otimes$-invertible.*

(b) *For any separated morphism of B-schemes of finite type $f: X\to S$, with S regular, and for any dualizing object U in $\mathrm{DM_h}(S,R)$, the object $f^!(U)$ is a dualizing object in $\mathrm{DM_h}(X,R)$.*

Proof. Consider separated morphism of B-schemes of finite type $f: X\to S$, with S regular. Then we claim that the object $f^!(R_S)$ is dualizing in $\mathrm{DM_h}(X,R)$. Indeed, by virtue of Corollary 5.4.11 and Lemma 5.8.11, we may assume that $R=\mathbf{Q}$ or $R=\mathbf{Z}/p\mathbf{Z}$ for some prime p. In the first case, this is already known (see [CD12, Theorems 15.2.4 and 16.1.2]). If $R=\mathbf{Z}/p\mathbf{Z}$, as, for any open immersion j, the functor j^* is symmetric monoidal and preserves internal Hom's, by virtue of Corolaries 4.5.6 and 5.4.6, we may assume that p is invertible in the residue fields of S and that we have equivalence of triangulated categories

$$\mathrm{D}^b_c(Y_{\acute{e}t},\mathbf{Z}/p\mathbf{Z})\simeq \mathrm{DM_h}(Y,\mathbf{Z}/p\mathbf{Z})$$

for any S-scheme of finite type, in a functorial way with respect to the six operations. In other words, this property boils down to the analogous result in classical étale cohomology (which, as this level of generality, has been proved by O. Gabber; see [ILO12]). This implies the theorem through classical and fomal arguments; see [CD12, Proposition 4.4.22]. □

5.9. **Completion and realisation.** In this section, we fix a prime ℓ and an integral flat $\mathbf{Z}$-algebra R.

Definition 5.9.1. Let X be a noetherian scheme.

We denote by $\mathrm{DM}_h(X,\hat{R}_\ell)$ the localizing subcategory of $\mathrm{DM}_h(X,R)$ generated by the objects of the form M/ℓ, for any constructible object M of $\mathrm{DM}_\mathrm{h}(X,R)$.

5.9.2. Recall from section 5.3 the following adjunctions of triangulated categories, expressing various change of coefficients:

$$\mathbf{L}\rho_\ell^* : \mathrm{DM}_h(X,R) \rightleftarrows \mathrm{DM}_h(X,R/\ell) : \rho_{\ell*},$$
$$\mathbf{L}\rho^* : \mathrm{DM}_h(X,R) \rightleftarrows \mathrm{DM}_h(X,R[\ell^{-1}]) : \rho_*,$$

where $\rho_\ell^*(M) = M/\ell$ and $\rho^*(M) = \mathbf{Z}[\ell^{-1}] \otimes M$. Note that, for any h-motive M in $\mathrm{DM}_h(X,R)$, the h-motive $\mathbf{Z}[\ell^{-1}] \otimes M$ is the homotopy colimit of the tower:

$$M \xrightarrow{\ell.1_M} M \xrightarrow{\ell.1_M} M \to \cdots \to M \xrightarrow{\ell.1_M} M \to \cdots$$

Moreover, the functor ρ_* is fully faithful, and identifies $\mathrm{DM}_h(X,R[\ell^{-1}])$ with the full subcategory of $\mathrm{DM}_h(X,R)$ whose objects are those on which the multiplication by ℓ is invertible. Such an object will be said *uniquely ℓ-divisible*.

Lemma 5.9.3. *For an object M of $\mathrm{DM}_\mathrm{h}(X,R)$, the following conditions are equivalent:*

(i) *M is uniquely ℓ-divisible;*
(ii) *$M/\ell \simeq 0$;*
(iii) *for any constructible object C of $\mathrm{DM}_\mathrm{h}(X,R)$, any map $C/\ell \to M$ is zero;*
(iv) *for any object C of $\mathrm{DM}_h(X,\hat{R}_\ell)$, any map from C to M is zero.*

Proof. The equivalence between conditions (i) and (ii) is trivial (in view of the distinguished triangle (5.3.4.b)), and the equivalence between conditions (iii) and (iv) is true by definition of $\mathrm{DM}_h(X,\hat{R}_\ell)$. The equivalence between conditions (ii) and (iii) comes from the fact that the objects of the form C/ℓ, with C constructible in $\mathrm{DM}_\mathrm{h}(X,R)$, form a generating family of the triangulated category $\mathrm{DM}_\mathrm{h}(X,\mathbf{Z}/\ell\mathbf{Z})$. □

5.9.4. We are thus in the *situation of the six gluing functors* as defined in [Nee01, 9.2.1]. This means that we have six functors:

$$\mathrm{DM}_h(X,\hat{R}_\ell) \underset{\hat{\rho}_{\ell*}}{\overset{\hat{\rho}_{\ell!}}{\underset{\xleftarrow{\hat{\rho}_\ell^*}}{\rightrightarrows}}} \mathrm{DM}_h(X,R) \underset{\rho^!}{\overset{\mathbf{L}\rho^*}{\underset{\xleftarrow{\rho_*}}{\rightrightarrows}}} \mathrm{DM}_h(X,R[\ell^{-1}]), \tag{5.9.4.a}$$

where $\hat{\rho}_{\ell!}$ denotes the inclusion functor, and that, for any h-motive in $\mathrm{DM}_h(X,R)$ we have functorial distinguished triangles

$$\hat{\rho}_{\ell!}\hat{\rho}_\ell^*(M) \xrightarrow{ad(\hat{\rho}_{\ell!},\hat{\rho}_\ell^*)} M \xrightarrow{ad'(\mathbf{L}\rho^*,\rho_*)} \rho_*\mathbf{L}\rho^*(M) \to M[1], \tag{5.9.4.b}$$

$$\rho_*\rho^!(M) \xrightarrow{ad(\rho_*,\rho^!)} M \xrightarrow{ad'(\hat{\rho}_\ell^*,\hat{\rho}_{l*})} \hat{\rho}_{\ell*}\hat{\rho}_\ell^*(M) \to M[1]. \tag{5.9.4.c}$$

Consider the obvious exact sequence of R-modules:

$$0 \to R \to R[\ell^{-1}] \to R[\ell^{-1}]/R \to 0.$$

It induces the following distinguished triangle in $\mathrm{DM}_h(X,R)$:

$$M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)[-1] \longrightarrow M \longrightarrow M \otimes^{\mathbf{L}} R[\ell^{-1}] \longrightarrow M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)$$

which is isomorphic to the triangle (5.9.4.b). In other words, we have the formulas:

$$\hat{\rho}_{\ell !}\hat{\rho}_\ell^*(M) = M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)[-1] \quad \text{and} \quad \rho_* \mathbf{L}\rho^*(M) = M[\ell^{-1}] = M \otimes \mathbf{Z}[\ell^{-1}].$$

5.9.5. Let M be a cofibrant object in the model category underlying $\mathrm{DM_h}(X,R)$. The h-motive M/ℓ^r is then represented by the complex of Tate spectra:

$$\mathrm{Coker}(M \xrightarrow{\ell^r.1_M} M).$$

Thus, we get a tower:

(5.9.5.a)
$$\begin{tikzcd} M \arrow[r, "\ell"] \arrow[d, "\ell"'] & M \arrow[r] \arrow[d, "\ell^2"] & \cdots \arrow[r] & M \arrow[r, "\ell"] \arrow[d, "\ell^r"'] & M \arrow[r] \arrow[d, "\ell^{r+1}"] & \cdots \\ M \arrow[r, equal] & M \arrow[r, equal] & \cdots \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots \end{tikzcd}$$

which defines a projective system $(M/l^r)_{r\in\mathbf{N}}$, and it makes sense to take its derived limit. This construction defines a triangulated functor

$$\mathrm{DM_h}(X,R) \to \mathrm{DM_h}(X,R)\,, \quad M \mapsto \mathbf{R}\varprojlim_r M/\ell^r\,.$$

Furthermore, the towers (5.9.5.a) define a natural transformation

(5.9.5.b)
$$\epsilon_\ell^M : M \to \mathbf{R}\varprojlim_{r\in\mathbf{N}} M/\ell^r.$$

Lemma 5.9.6. *For any h-motive M in $\mathrm{DM}_h(X,R)$, we have a canonical isomorphism:*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(R[\ell^{-1}]/R, M)[1] \simeq \mathbf{R}\varprojlim_{r\in\mathbf{N}} M/\ell^r.$$

Proof. We have

$$\mathbf{Z}[\ell^{-1}]/\mathbf{Z} = \varinjlim_r \mathbf{Z}/\ell^r\mathbf{Z},$$

from which we get that

$$R \otimes \mathbf{Z}[\ell^{-1}]/\mathbf{Z} = \varinjlim_r R/\ell^r R\,.$$

As this colimit is filtering, this is in fact an homotopy colimit, and we conclude from the canonical isomorphisms $\mathbf{R}\underline{\mathrm{Hom}}(\mathbf{Z}/\ell^r\mathbf{Z}, M)[1] \simeq M/\ell^r$. □

Definition 5.9.7. For any h-motive M in $\mathrm{DM}_h(X,R)$, we define the *ℓ-completion* of M as the h-motive:

$$\hat{M}_\ell := \mathbf{R}\varprojlim_{n\in\mathbf{N}} M/\ell^r.$$

We say that M is *ℓ-complete* if the map $\epsilon_l^M : M \to \hat{M}_\ell$ defined above is an isomorphism.

According to Lemma 5.9.6 and Paragraph 5.9.4, the triangle (5.9.4.c) can be identified to the triangle:

$$\mathbf{R}\underline{\mathrm{Hom}}(R[\ell^{-1}], M) \longrightarrow M \xrightarrow{\epsilon_\ell^M} \hat{M} \xrightarrow{+1}$$

Note in particular the following well known fact (see for instance [DG02]).

Proposition 5.9.8. *Let M be an h-motive in $\mathrm{DM_h}(X,R)$. Then the following conditions are equivalent:*

(i) *M belongs to $\mathrm{DM_h}(X,\hat{R}_\ell)$.*
(ii) *M is ℓ-complete.*

(iii) *M is left orthogonal to uniquely ℓ-divisible objects in* $\mathrm{DM_h}(X,R)$.

Lemma 5.9.6 readily implies the following computation, which means (at least when ℓ is prime to the residue chracteristics of X), in view of the equivalences $\mathrm{DM_h}(X,R/\ell^r) \simeq \mathrm{D}(X_{\text{ét}},R/\ell^r)$, that the category $\mathrm{DM_h}(X,\hat{R}_\ell)$ is a categorical incarnation of continuous étale cohomology in the sense of Jannsen [Jan88].

Proposition 5.9.9. *For any objects M and N in* $\mathrm{DM_h}(X,\hat{R}_\ell)$*, we have*

$$\mathbf{R}\mathrm{Hom}_{\mathrm{DM_h}(X,\hat{R}_\ell)}(M,N) \simeq \mathbf{R}\varprojlim_r \mathbf{R}\mathrm{Hom}_{\mathrm{DM_h}(X,R/\ell^r)}(M/\ell^r,N/\ell^r).$$

5.9.10. The right adjoints f_*, $\mathbf{R}\underline{\mathrm{Hom}}$ of the triangulated premotivic category $\mathrm{DM}_h(-,R)$ commutes with homotopy limits. Moreover, Proposition 5.3.5 shows they preserve ℓ-complete objects.

On the other hand, for any morphism of scheme $f:Y\to X$, and smooth morphism $p:X\to S$ and any ℓ-complete h-motives M, N, we put:

$$\hat{f}^*(M)=\widehat{f^*(M)}_\ell\ ,\quad \hat{p}_\sharp(M)=\widehat{p_\sharp(M)}_\ell\ ,\quad M\hat{\otimes}N=(\widehat{M\otimes N})_\ell\ .$$

This defines a structure of a premotivic triangulated category on $\mathrm{DM_h}(-,\hat{R}_\ell)$, the right adjoints being induced their counterpart in $\mathrm{DM_h}(-,R)$.

According to these definitions, we get a premotivic adjunction:

$$\hat{\rho}_\ell^*:\mathrm{DM}_h(-,R)\rightleftarrows \mathrm{DM}_h(-,\hat{R}_\ell):\hat{\rho}_{\ell *}. \tag{5.9.10.a}$$

Moreover, $\hat{\rho}_\ell^*$ obviously commutes with f_* and $\underline{\mathrm{Hom}}$.

Taking into account Proposition 5.8.9, Corollary 5.3.11, Theorem 5.5.2, and Lemma 5.9.6, we thus obtain:

Theorem 5.9.11. *The triangulated premotivic category $\mathrm{DM}_h(-,\hat{R}_l)$ satisfies the Grothendieck six functors formalism (Def. A.1.10) and the absolute purity property (Def. A.2.9) over quasi-excellent noetherian schemes. The premotivic morphism $\hat{\rho}_\ell^*$ defined above commutes with the six operations (Def. A.1.17).*

Remark 5.9.12. Note that, by virtue of Theorem 5.4.5, if we perform this ℓ-completion procedure to $\mathrm{DM}^{eff}_{\text{ét}}(X,R)$ or $\mathrm{DM}^{eff}_{\mathrm{h}}(X,R)$, this leads to the same category $\mathrm{DM}_h(-,\hat{R}_\ell)$.

Definition 5.9.13. Let X be any scheme. One defines the category $\mathrm{DM}_{\mathrm{h},gm}(X,\hat{R}_\ell)$ of *geometric ℓ-adic* h-*motives* as the thick triangulated subcategory of $\mathrm{DM_h}(X,\hat{R}_\ell)$ generated by h-motives of the form $\hat{R}^{\mathrm{h}}_S(X)(n)$ for X/S smooth and $n\in\mathbf{Z}$. An object M of $\mathrm{DM_h}(X,\hat{R}_\ell)$ is said to be *constructible* if, M/ℓ is constructible in $\mathrm{DM_h}(X,R/\ell)$. We write $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$ for the thick subcategory of $\mathrm{DM_h}(X,\hat{R}_\ell)$ generated by constructible ℓ-adic motives. We thus have a natural inclusion

$$\mathrm{DM}_{\mathrm{h},gm}(X,\hat{R}_\ell)\subset \mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell).$$

Remark 5.9.14. The notion of constructible ℓ-adic motive corresponds to what is usually called (bounded complex of) constructible ℓ-adic sheaves, while geometric ℓ-adic h-motives correspond to (bounded complex of) constructible ℓ-adic sheaves *of geometric origin*.

Remark 5.9.15. It is clear that $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$ is closed under the six operations in $\mathrm{DM_h}(X,\hat{R}_\ell)$: this readily follows from Corollary 5.8.9 in the case of $R=\mathbf{Z}/\ell\mathbf{Z}$: indeed, the functor

$$\mathrm{DM_h}(X,\hat{R}_\ell)\to \mathrm{DM_h}(X,R/\ell)\simeq \mathrm{D}(X_{\text{ét}},R/\ell)\ ,\quad M\mapsto M/\ell$$

is conservative and preserves the six operations as well as constructible objects (by definition). Note also that an object M of $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ is constructible if and only if M/ℓ^r is constructible in $\mathrm{DM}_{\mathrm{h}}(X,R/\ell^r)$ for any $r \geq 1$.

Theorem 5.9.16. *The ℓ-adic realization functor of Theorem 5.9.11 sends constructible objects to geometric ones. Moreover, the six operations preserve geometric objects in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$.*

Proof. The first assertion is obvious. To prove that the subcategory $\mathrm{DM}_{\mathrm{h},gm}(X,\hat{R}_\ell)$ is closed under the six operations in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$, it is sufficient check what happens on objects of the form $\hat{M}_\ell$ with M constructible in $\mathrm{DM}_{\mathrm{h}}(X,R)$. But then, the fact that the ℓ-adic realization functor preserves the six operations on the nose means that they preserve the class of these objects in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$. □

5.9.17. Let ℓ be a prime, and S a noetherian scheme with residue characteristics prime to ℓ, and such that, for any constructible sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules F on $S_{\text{ét}}$, the cohomology groups $H^i_{\text{ét}}(S,F)$ are finite. Then, for any S-scheme of finite type X, one can define, following Beilinson, Bernstein and Deligne [BBD82], the *triangulated category of constructible ℓ-adic sheaves* as the following 2-limit of derived categories of constructible sheaves:

$$\mathrm{D}^b_c(X,\mathbf{Z}_\ell) = 2\text{-}\varprojlim_r \mathrm{D}^b_c(X,\mathbf{Z}/\ell^r\mathbf{Z}).$$

On the other hand, we have an obvious family of triangulated functors

$$\mathrm{DM}_{\mathrm{h},c}(X,\hat{\mathbf{Z}}_\ell) \to \mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z}/\ell^r\mathbf{Z})\,, \quad M \mapsto \mathbf{Z}/\ell^r\mathbf{Z} \otimes^{\mathbf{L}} M$$

which, together with the equivalences of categories given by Corollary 5.4.6,

$$\mathrm{D}^b_c(X,\mathbf{Z}/\ell^r\mathbf{Z}) \simeq \mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z}/\ell^r\mathbf{Z}),$$

induce a triangulated functor

$$\mathrm{DM}_{\mathrm{h},c}(X,\hat{\mathbf{Z}}_\ell) \to \mathrm{D}^b_c(X,\mathbf{Z}_\ell) \tag{5.9.17.a}$$

Proposition 5.9.18. *Under the assumptions of 5.9.17, the functor* (5.9.17.a) *is an equivalence of categories.*

Proof. Let M and N be two objects of $\mathrm{DM}_{\mathrm{h},c}(X,\hat{\mathbf{Z}}_\ell)$. By virtue of Proposition 5.9.8, we have

$$N = \mathbf{R}\varprojlim_r N/\ell^r\,.$$

Moreover, by assumption, for any $r \geq 1$, the groups $\mathrm{Hom}(M/\ell^r,N/\ell^r)$ are finite, and thus, for any integer i, we have

$$\mathrm{Hom}(M,N[i]) = H^i(\mathbf{R}\varprojlim_r \mathbf{R}\mathrm{Hom}(M,N/\ell^r)) \simeq \varprojlim_r \mathrm{Hom}(M,N/\ell^r[i])\,.$$

The fully faithfulness of the functor (5.9.17.a) readily follows from this computation. Let F be an object of $\mathrm{D}^b_c(X,\mathbf{Z}_\ell)$, that is a collection of objects F_r in $\mathrm{D}^b_c(X,\mathbf{Z}/\ell^r\mathbf{Z})$, together with isomorphisms

$$u_r : \mathbf{Z}/\ell^r\mathbf{Z} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{r+1}\mathbf{Z}} F_{r+1} \simeq F_r$$

for each $r \geq 1$. Such data can be lifted into a collection (E_r, v_r), where E_r is a complex of sheaves of $\mathbf{Z}/\ell^r\mathbf{Z}$-modules on $X_{\text{ét}}$, and

$$v_r : \mathbf{Z}/\ell^r\mathbf{Z} \otimes_{\mathbf{Z}/\ell^{r+1}\mathbf{Z}} E_{r+1} \to E_r$$

is a $\mathbf{Z}/\ell^r\mathbf{Z}$-linear morphism of complexes of sheaves for each $r \geq 1$, such that $E_r \simeq F_r$ in $\mathrm{D}^b_c(X, \mathbf{Z}/\ell^r\mathbf{Z})$, and such that the canonical map

$$\mathbf{Z}/\ell^r\mathbf{Z} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{r+1}\mathbf{Z}} E_{r+1} \to \mathbf{Z}/\ell^r\mathbf{Z} \otimes_{\mathbf{Z}/\ell^{r+1}\mathbf{Z}} E_{r+1} \to E_r$$

coincides with the given isomorphism u_r under these identifications. Applying the functor α^* (5.2.1.a), this defines similar data $(\alpha^*(E_r), \alpha^*(v_r))$ in the category of complexes of sheaves over the h-site of X. We may assume that each sheaf E_r if flat over $\mathbf{Z}/\ell^r\mathbf{Z}$ (by choosing them cofibrant for the projective model structure, for instance), in which case the maps v_r already are quasi-isomorphisms. Applying the infinite suspension functor Σ^∞, finally leads to a diagram of Tate spectra, and we can define

$$E = \mathbf{R}\varprojlim_r \Sigma^\infty(\alpha^*(E_r)).$$

Note that, for any integer $r \geq 1$, we have $E/\ell^r \simeq \Sigma^\infty(\alpha^*(E_r))$ in $\mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Z}/\ell^r\mathbf{Z})$. We thus see through the equivalences

$$\mathrm{D}^b_c(X, \mathbf{Z}/\ell^r\mathbf{Z}) \simeq \mathrm{DM}^{eff}_{\mathrm{h},c}(X, \mathbf{Z}/\ell^r\mathbf{Z}) \simeq \mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Z}/\ell^r\mathbf{Z})$$

that the functor (5.9.17.a) sends E to an object isomorphic to F. □

5.9.19. Recall that T. Ekedahl has constructed in [Eke90, Th. 6.3] a triangulated monoidal category: $\mathrm{D}^b_c(X - \mathbf{Z}.)$ of ℓ-adic constructible systems over a separated S-scheme X of finite type, assuming suitable technical conditions on S.

Using Corollary 5.4.6 and Proposition 5.9.9, one can construct an equivalence of categories

$$\mathrm{D}^b_c(X - \mathbf{Z}.) \to \mathrm{DM}_{h,c}(X, \hat{\mathbf{Z}}_\ell)$$

Moreover, using point (3) of *loc. cit.*, one can see from the above definitions that this functor commutes with the 6 operations on the category of separated S-schemes of finite type.

But we will not do this here. Instead, we define

$$\mathrm{D}^b_c(X, \mathbf{Z}_\ell) := \mathrm{DM}_{h,c}(X, \hat{\mathbf{Z}}_\ell)$$

for any noetherian scheme X. We also define the category of $\mathbf{Q}_\ell$-sheaves over X

$$\mathrm{D}^b_c(X, \mathbf{Q}_\ell) = \mathrm{DM}_{h,c}(X, \hat{\mathbf{Z}}_\ell) \otimes \mathbf{Q}$$

as the $\mathbf{Q}$-linearisation of the triangulated category $\mathrm{DM}_{h,c}(X, \hat{\mathbf{Z}}_\ell)$. Both $\mathrm{D}^b_c(-, \mathbf{Z}_\ell)$ and $\mathrm{D}^b_c(-, \mathbf{Q}_\ell)$ are motivic categories which satisfy the absolute purity property (at least when restricted to quasi-excellent noetherian schemes of finite dimension).

5.9.20. As a final result, taking into account the fact the $\mathbf{Q}$-localization functor is well behaved for h-motives (Corollary 5.3.11), we have a canonical identification, for any quasi-excellent noetherian scheme of finite dimension:

$$\big(\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}) \otimes \mathbf{Q}\big)^\sharp \simeq \mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Q}).$$

We thus obtain straight away the following result.

Theorem 5.9.21. *The functor* $\hat{\rho}^*_\ell$ (5.9.10.a) *induces, for any noetherian scheme of finite dimension X, a triangulated monoidal functor:*

$$\mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Q}) \to \mathrm{D}^b_c(X, \mathbf{Q}_\ell)^\sharp$$

(where $\mathrm{D}^b_c(X, \mathbf{Q}_\ell)^\sharp$ is the idempotent completion of the triangulated category $\mathrm{D}^b_c(X, \mathbf{Q}_\ell)$). This functor is compatible with the 6 operations (when one restricts our attention to quasi-excellent schemes and morphisms of finite type between them).

APPENDIX A. RECALL AND COMPLEMENT ON PREMOTIVIC CATEGORIES

A.1. **Premotivic categories and morphisms.** The following definition is a summary of the definitions in [CD09, sec. 1].

Definition A.1.1. Let $\mathscr{P}$ be one of the classes: Ét, Sm, $\mathscr{S}^{ft}$.

A *triangulated (*resp. *abelian)* $\mathscr{P}$*-premotivic category* $\mathscr{M}$ is a fibred category over Sch satisfying the following properties:

(1) For any scheme S, $\mathscr{M}_S$ is a well generated triangulated (resp. abelian Grothendieck) category with a closed monoidal structure.[7]
(2) For any morphism of schemes f, the functor f^* is triangulated (resp. additive), monoidal and admits a right adjoint denoted by f_*.
(3) For any morphism p in $\mathscr{P}$, the functor p^* admits a left adjoint denoted by $p_\sharp$.
(4) $\mathscr{P}$*-base change*: For any cartesian square

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] & S \end{tikzcd}$$

there exists a canonical isomorphism: $Ex(\Delta^*_\sharp) : q_\sharp g^* \to f^* p_\sharp$.
(5) $\mathscr{P}$*-projection formula*: For any morphism $p : T \to S$ in $\mathscr{P}$, and any object (M,N) of $\mathscr{M}_T \times \mathscr{M}_S$, there exists a canonical isomorphism:

$$Ex(p^*_\sharp, \otimes) : p_\sharp(M \otimes_T p^*(N)) \to p_\sharp(M) \otimes_S N.$$

When $\mathscr{P} = Sm$, we say simply *premotivic* instead of Sm-premotivic. Objects of $\mathscr{M}$ are generically called *premotives*.

Remark A.1.2. The isomorphisms appearing in properties (4) and (5) are particular instances of what is generically called an *exchange transformation* in [CD09].

Example A.1.3. Let $\mathscr{P}$ be one of the classes: Ét, Sm, $\mathscr{S}^{ft}$.

Then the categories $\mathrm{Sh}_{\text{ét}}(\mathscr{P}_S, R)$ (resp. $\mathrm{Psh}(\mathscr{P}_S, R)$) of étale sheaves (resp. presheaves) of R-modules over $\mathscr{P}_S$ for various base schemes S form the fibers of an abelian premotivic category (see [CD12, Ex. 5.1.1]).

Moreover, the derived categories $\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}_S, R))$ (resp. $\mathrm{D}(\mathrm{Psh}(\mathscr{P}_S, R))$) for various schemes S form the fibers of a canonical triangulated premotivic category (see [CD12, Def. 5.1.17]).

A.1.4. Consider a premotivic triangulated category $\mathscr{T}$.

Given any smooth morphism $p : X \to S$, we define following Voevodsky the *(homological) premotive* associated with X/S as the object: $M_S(X) := p_\sharp(\mathbb{1}_X)$. Then M_S is a covariant functor.

Let $p : \mathbf{P}^1_S \to S$ be the canonical projection. We define the *Tate premotive* as the kernel of the map $p_* : M_S(\mathbf{P}^1_S) \to \mathbb{1}_S$ shifted by -2. Given an integer n and an object M of $\mathscr{T}$, we define the *n-th Tate twist* $M(n)$ of M as the n-th tensor power of M by the object $\mathbb{1}(1)$ – allowing negative n if $\mathbb{1}(1)$ is $\otimes$-invertible.

We associate to $\mathscr{T}$ a bigraded cohomology theory on Sch:

$$H^{i,n}_{\mathscr{T}}(S) := \mathrm{Hom}_{\mathscr{T}}(\mathbb{1}_S, \mathbb{1}_S(n)[i]).$$

[7] In the triangulated case, we require that the bifunctor $\otimes$ is triangulated in each variable.

One can isolate the following basic properties of $\mathscr{T}$ (see [CD12]).

Definition A.1.5. Consider the notations above. One introduces the following properties of the premotivic triangulated category $\mathscr{T}$:

(1) *Homotopy property*.– For any scheme S, the canonical projection of the affine line over S induces an isomorphism $M_S(\mathbf{A}^1_S) \to \mathbb{1}_S$.
(2) *Stability property*.– The Tate premotive $\mathbb{1}(1)$ is $\otimes$-invertible.
(3) *Orientation*.– An *orientation of* $\mathscr{T}$ is natural transformation of contravariant functors
$$c_1 : \mathrm{Pic} \to H^{2,1}$$
(not necessarily additive).[8]
When $\mathscr{T}$ is equiped with an orientation one says $\mathscr{T}$ is *oriented*.

A.1.6. Recall that a cartesian functor $\varphi^* : \mathscr{T} \to \mathscr{T}'$ between fibred categories over *Sch* is the following data:

- for any base scheme S in *Sch*, a functor $\varphi^*_S : \mathscr{T}(S) \to \mathscr{T}'(S)$.
- for any morphism $f : T \to S$ in *Sch*, a natural isomorphism $c_f : f^*\varphi^*_S \xrightarrow{\sim} \varphi^*_T f^*$ satisfying the cocycle condition.

The following definition is a particular case of [CD12, Def. 1.4.6]:

Definition A.1.7. Let $\mathscr{P}$ be one of the classes: Ét, *Sm*, $\mathscr{S}^{ft}$.

A morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of *triangulated (*resp. *abelian) $\mathscr{P}$-premotivic categories* is a cartesian functor satisfying the following properties:

(1) For any scheme S, φ^*_S is triangulated (resp. additive), monoidal and admits a right adjoint denoted by φ_{S*}.
(2) For any morphism $p : T \to S$ in $\mathscr{P}$, there exists a canonical isomorphism: $Ex(p_\sharp, \varphi^*) : p_\sharp \varphi^*_T \to \varphi^*_S p_\sharp$.

Sometimes, we refer to such a morphism as the *premotivic adjunction*:
$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_*.$$
A *sub-$\mathscr{P}$-premotivic triangulated (*resp. *abelian) category* $\mathscr{M}_0$ of $\mathscr{M}$ is a full triangulated (resp. additive) sub-category of $\mathscr{M}$ equipped with a $\mathscr{P}$-premotivic structure such that the inclusion $\mathscr{M}_0 \to \mathscr{M}$ is a morphism of $\mathscr{P}$-premotivic categories.

Remark A.1.8. Given a morphism of triangulated premotivic categories
$$\varphi^* : \mathscr{T} \to \mathscr{T}',$$
any orientation of $\mathscr{T}$ induces a canonical orientation of $\mathscr{T}'$. Indeed, we deduce from the preceding that for any scheme X, the functor φ^*_X induces a morphism
$$H^{2,1}_{\mathscr{T}}(X) \to H^{2,1}_{\mathscr{T}'}(X)$$
contravariantly natural in X.

Example A.1.9. Consider the notations of Example A.1.3

Recall from [CD12, Def. 5.2.16] the $\mathbf{A}^1$-localization $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathscr{P},R))$ of $\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P},R))$, which which is a $\mathscr{P}$-fibred category equipped with a localization morphism
$$\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P},R)) \to \mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathscr{P},R))$$
and satisfying the homotopy property.

[8] However, the orientations which appear in this article are always additive.

When $\mathscr{P} = Sm$, we will put: $\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(S,R) = \mathrm{D}^{eff}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm_S,R))$.

The main properties of a triangulated premotivic category can be summarized in the so called Grothendieck 6 functors formalism:

Definition A.1.10. A triangulated premotivic category $\mathscr{T}$ satisfies *Grothendieck 6 functors formalism* if it satisfies the stability property and for any separated morphism of finite type $f : Y \to X$ in Sch, there exists a pair of adjoint functors

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

such that:

(1) There exists a structure of a covariant (resp. contravariant) 2-functor on $f \mapsto f_!$ (resp. $f \mapsto f^!$).
(2) There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.
(3) For any smooth morphism $f : X \to S$ in Sch of relative dimension d, there are canonical natural isomorphisms

$$\mathfrak{p}_f : f_\sharp \longrightarrow f_!(d)[2d]$$
$$\mathfrak{p}'_f : f^* \longrightarrow f^!(-d)[-2d]$$

which are dual to each other.
(4) For any cartesian square in Sch:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Delta"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

such that f is separated of finite type, there exist natural isomorphisms

$$g^* f_! \xrightarrow{\sim} f'_! g'^*,$$
$$g'_* f'^! \xrightarrow{\sim} f^! g_*.$$

(5) For any separated morphism of finite type $f : Y \to X$, there exist natural isomorphisms

$$Ex(f_!^*, \otimes) : (f_! K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L),$$
$$\underline{\mathrm{Hom}}_X(f_!(L), K) \xrightarrow{\sim} f_* \underline{\mathrm{Hom}}_Y(L, f^!(K)),$$
$$f^! \underline{\mathrm{Hom}}_X(L, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_Y(f^*(L), f^!(M)).$$

(6) For any closed immersion $i : Z \to S$ with complementary open immersion j, there exists distinguished triangles of natural transformations as follows:

$$j_! j^! \xrightarrow{\alpha'_j} 1 \xrightarrow{\alpha_i} i_* i^* \xrightarrow{\partial_i} j_! j^![1]$$
$$i_! i^! \xrightarrow{\alpha'_i} 1 \xrightarrow{\alpha_j} j_* j^* \xrightarrow{\tilde{\partial}_i} i_! i^![1]$$

where $\alpha'_?$ (resp. $\alpha_?$) denotes the counit (resp. unit) of the relevant adjunction.

A.1.11. In [CD12], we have studied some of these properties axiomatically, introducing the following definitions:

- Given a closed immersion i, the fact i_* is conservative and the existence of the first triangle in (6) is called the *localization property with respect to* i.
- The conjunction of properties (2) and (3) gives, for a smooth proper morphism f, an isomorphism $\mathfrak{p}_f : f_\sharp \to f_*(d)[2d]$. Under the stability and weak localization properties, when such an isomorphism exists, we say that f is $\mathscr{T}$*-pure* (or simply *pure* when $\mathscr{T}$ is clear).[9]

Definition A.1.12. Consider the notations an assumptions above.

We say that $\mathscr{T}$ satisfies the *localization property* (resp. *weak localization property*) if it satisfies the localization property with respect to any closed immersion i (resp. which admits a smooth retraction).

We say that $\mathscr{T}$ satisfies the *purity property* (resp. *weak purity property*) if for any smooth proper morphism f (resp. for any scheme S and integer $n > 0$, the projection $p : \mathbf{P}^n_S \to S$) is $\mathscr{T}$-pure.

Building on the construction of Deligne of $f_!$ and on the work of Ayoub on cross functors, we have obtained in [CD12, th. 2.4.50] the following theorem which is little variation on a theorem of Ayoub:

Theorem A.1.13. *The following conditions on a well generated triangulated premotivic category $\mathscr{T}$ equipped with an orientation and satisfying the homotopy property are equivalent:*

(i) *$\mathscr{T}$ satisfies Grothendieck 6 functors formalism.*
(ii) *$\mathscr{T}$ satisfies the stability and localization properties.*

Remark A.1.14. In fact, J. Ayoub in [Ayo07] proves this result with the following notable differences:

- One has to restrict to a category of quasi-projective schemes over a scheme which admits an ample line bundle.
- The questions of orientation are not treated in *op. cit.*: this means one has to replace the Tate twist in property (3) above by the tensor product with a *Thom space*.
- The theorem of Ayoub is more general in the sense that it does not require an orientation on the category $\mathscr{T}$. In particular, it applies to the stable homotopy category of schemes, which does not admit an orientation.

Recall the following definition from [CD12]:

Definition A.1.15. A triangulated premotivic category $\mathscr{T}$ which satisfies the stability and localization properties, and in which the functor $f^!$ exists for any proper morphism f in *Sch*, is called a triangulated motivic category.

A.1.16. Consider an adjunction

$$\varphi^* : \mathscr{T} \rightleftarrows \mathscr{T}' : \varphi_*$$

of triangulated premotivic categories which satisfies Grothendieck 6 functors formalism. Then it is proved in [CD12] that φ^* commutes with $f_!$ for f separated of finite type. In fact, φ^* commutes with the left adjoint of the 6 functors formalism while φ_* commutes with the right adjoint functors.

[9] In fact, the isomorphism $\mathfrak{p}_f$ is canonical up to the choice of an orientation of $\mathscr{T}$. Moreover, we will define explicitely this isomorphism in the case where we need it – see (4.2.5.a).

On the other hand, there are canonical exchange transformations:

$$\begin{aligned} &\varphi^* f_* \to f_* \varphi^*, f \text{ morphism in } Sch,\\ \text{(A.1.16.a)} \qquad &\varphi^* f^! \to f^! \varphi^*, f \text{ separated morphism of finite type in } Sch,\\ &[\varphi^* \underline{\mathrm{Hom}}(-,-)] \longrightarrow [\underline{\mathrm{Hom}}(\varphi^*(-), \varphi^*(-))]. \end{aligned}$$

Definition A.1.17. In the above assumptions, one says the morphism φ^* *commutes with the 6 operations* if the exchange transformations (A.1.16.a) are all isomorphisms.

If $\mathscr{T}$ is a sub-premotivic triangulated category of $\mathscr{T}'$, one simply says $\mathscr{T}$ is *stable by the 6 operations* if the inclusion commutes with the 6 operations.

For example, if φ^* is an equivalence of premotivic triangulated categories, then it commutes with the 6 operations.

A.2. **Complement: the absolute purity property.** In this section, we consider a triangulated premotivic category $\mathscr{T}$ which satisfies the hypothesis and equivalent conditions of Theorem A.1.13. We assume in addition that the motives of the form $M_S(X)(i)$ for a smooth S-scheme X and a Tate twist $i \in \mathbf{Z}$ form a family of generators of the category $\mathscr{T}(S)$.

A.2.1. As usual, a closed pair is a pair of schemes (X,Z) such that Z is a closed subscheme of X. We will consider abusively that to give such a closed pair is equivalent to give a closed immersion $i: Z \to X$. We will say (X,Z) is regular when i is regular.

A (cartesian) morphism of closed pairs $(f,g):(Y,T) \to (X,Z)$ is a cartesian square of schemes:

(A.2.1.a)
$$\begin{tikzcd} T \arrow[r, hook, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

We will usually denote it by f instead of (f,g).

Note the preceding diagram induces a unique map $C_T Y \to g^{-1}(C_Z X)$ on the underlying normal cones. We say f (or the above square) is *transversal* when this map is an isomorphism.

Definition A.2.2. Let (X,Z) be a closed pairs and $i: Z \to X$ be the canonical inclusion. For any pair of integers (n,m), we define the cohomology of X with support in Z as:

$$H^{n,m}_Z(X) := \mathrm{Hom}_{\mathscr{T}(S)}(i_*(\mathbb{1}_Z), \mathbb{1}_S(m)[n]).$$

Equivalently,

$$H^{n,m}_Z(X) = \mathrm{Hom}_{\mathscr{T}(Z)}(\mathbb{1}_Z, i^!(\mathbb{1}_S)(m)[n]). \qquad \text{(A.2.2.a)}$$

Moreover, using the first localization triangle for $\mathscr{T}$ with respect to i (point (6), Def. A.1.10), we obtain it is contravariantly functorial with respect to morphism of closed pairs.

Remark A.2.3. (1) Using this localization triangle, this cohomology can be inserted in the usual localization long exact sequence (the twist m being the same for each group).

(2) Consider a morphism of closed pairs $f:(Y,T)\to(X,Z)$ defined by a cartesian square of the form (A.2.1.a). Using point (4) of Definition A.1.10 applied to this square, we can define the following *exchange transformation*:

(A.2.3.a) $$Ex^{*!}:g^*i^! \xrightarrow{ad(f_*,f^*)} g^*i^!f_*f^* \xrightarrow{\sim} g^*g_*k^!f^* \xrightarrow{ad'(g_*,g^*)} k^!f^*.$$

One can check that the functoriality property of $H^{**}_Z(X)$ is given by associating to a morphism $\rho:\mathbb{1}_Z\to i^!(\mathbb{1}_Z)(i)[n]$ the composite map:

$$\mathbb{1}_T \xrightarrow{g^*(\rho)} g^*i^!(\mathbb{1}_Z)(i)[n] \xrightarrow{Ex^{*!}} k^!(\mathbb{1}_T)(i)[n]$$

through the identification (A.2.2.a).

According to formula (A.2.2.a), the bigraded cohomology group $H^{**}(X)$ admits a structure of a bigraded module over the cohomology ring $H^{**}(Z)$. According to the preceding remark, this module structure in compatible with pullbacks.

Definition A.2.4. Let (X,Z) be a regular closed pair of codimension c. A *fundamental class* of Z in X is an element

$$\eta_X(Z)\in H^{2c,c}_Z(X)$$

which is a base of the $H^{**}(Z)$-module $H^{**}_Z(X)$.

In other words, the canonical map:

(A.2.4.a) $$H^{**}(Z)\to H^{**}_Z(X)\,,\quad \lambda\mapsto\lambda.\eta_X(Z)$$

is an isomorphism. Note that if such a fundamental class exists, it is unique up to an invertible element of $H^{00}(Z)$.

Proposition A.2.5. *Consider a regular closed immersion $i:Z\to X$ of codimension c and a morphism in $\mathscr{T}(Z)$:*

$$\eta_X(Z):\mathbb{1}_Z\to i^!(\mathbb{1}_X)(c)[2c].$$

The following conditions are equivalent:

(i) *The map $\eta_X(Z)$ is an isomorphism.*

(ii) *For all smooth morphism $f:Y\to X$, the cohomology class $f^*(\eta_X(Z))$ in $H^{2c,c}_{f^{-1}(T)}(Y)$ is a fundamental class.*

Proof. We first remark that for any smooth X-scheme Y, $T=Y\times_X Z$, and for any couple of integers $(n,r)\in\mathbf{Z}^2$, the map induced by $\eta_X(Z)$:

$$\mathrm{Hom}(M_Z(T)(-r)[-n],\mathbb{1}_Z)\to\mathrm{Hom}(M_Z(T)(-r)[-n],i^!(\mathbb{1}_X)(c)[2c])$$

is isomorphic to the map

$$H^{n,r}(T)\to H^{n,r}_T(Y),\lambda\mapsto\lambda.\eta_T(Y).$$

Then the equivalence between (i) and (ii) follows from the fact the family of motives of the form $M_Z(Y\times_Z X)(-r)[-n]$ generates the category $\mathscr{T}(Z)$ because:

- We have assumed $\mathscr{T}$ it is generated by Tate twist as a triangulated premotivic category.
- i^* is essentially surjective according to the localization property.

□

Using the arguments[10] of [Dég08], one obtains that the orientation $c_1 : \mathrm{Pic} \to H^{2,1}$ can be extended canonically to a full theory of Chern classes and deduced the projective bundle formula. One gets in particular, following Paragraph 4.4 of *loc. cit.*:

Proposition A.2.6. *Let E be a vector bundle over a scheme X, $s : X \to E$ the zero section. Then s admits a canonical*[11] *fundamental class.*

This is the *Thom class* defined in *loc. cit.* In what follows we will denote it by $\mathrm{th}(E)$, as an element of $H^{2c,c}_X(E)$.

A.2.7. Let (X,Z) be a closed pair with inclusion $i : Z \to X$. Assume i is a regular closed immersion of codimension c.

Following the classical construction, one define the *deformation space* $D_Z X$ attached to (X,Z) as the complement of the blow-up $B_Z(X)$ in $B_Z(\mathbf{A}^1_X)$. Note it contains $\mathbf{A}^1_Z$ as a closed subscheme.

This space is fibered over $\mathbf{A}^1$, with fiber over 1 (resp. 0) being the scheme X (resp. the normal bundle $N_Z X$). In particular, we get morphisms of closed pairs:

$$(X,Z) \xrightarrow{d_1} (D_Z X, \mathbf{A}^1_Z) \xleftarrow{d_0} (N_Z X, Z) \tag{A.2.7.a}$$

where d_0 (resp. d_1) means inclusion of the fiber over 0 (resp. 1). It is important to note that d_0 and d_1 are transversal.

For the next statement, we denote by $\mathscr{P}_{\mathrm{reg}}$ the class of closed pairs (X,Z) in *Sch* such that X and Z are regular.

Theorem A.2.8. *The following conditions are equivalent:*

(i) *There exists a family*

$$\big(\eta_X(Z)\big)_{(X,Z)\in\mathscr{P}_{\mathrm{reg}}}$$

such that:

- *For any closed pair (X,Z), $\eta_X(Z)$ is a fundamental class of (X,Z).*
- *For any transversal morphism $f : (Y,T) \to (X,Z)$ of closed pairs in $\mathscr{P}_{\mathrm{reg}}$, $f^*\eta_X(Z) = \eta_Y(T)$.*

(ii) *For any closed pair (X,Z) in $\mathscr{P}_{\mathrm{reg}}$, the deformation diagram* (A.2.7.a) *induces isomorphisms of bigraded cohomology groups:*

$$H^{**}_Z(X) \xleftarrow{d_1^*} H^{**}_{\mathbf{A}^1_Z}(D_Z X) \xrightarrow{d_0^*} H^{**}_Z(N_Z X)$$

Proof. The fact (i) implies (ii) follows from the homotopy property of $\mathscr{T}$, using the isomorphism of type (A.2.4.a) and the fact the morphisms of closed pairs d_0 and d_1 are transversal.

Reciprocally, given the isomorphisms which appear in (ii), one can put $\eta_X(Z) = d_1^*(d_0^*)^{-1}(\mathrm{th}(N_Z X))$, using Proposition A.2.6. This is a fundamental class for (X,Z) using once again the homotopy property for $\mathscr{T}$. The fact these classes are stable by transversal base change follows from the functoriality of the deformation diagram (A.2.7.a) with respect to transversal morphisms. $\square$

Definition A.2.9. We will say that $\mathscr{T}$ satisfies the *absolute purity property* if the equivalent properties of the preceding propositions are satisfied.

[10] In fact, if $\mathscr{T}$ is equipped with a premotivic morphism $\mathrm{D}(\mathrm{PSh}(-,R)) \to \mathscr{T}$, one can readily apply all the results of [Dég08] to the category $\mathscr{T}(S)$ for any fixed base scheme S. All the premotivic triangulated categories considered in this paper will satisfy this hypothesis.

[11] Depending only on the orientation c_1 of $\mathscr{T}$.

Example A.2.10.
(1) The motivic category of Beilinson motives DM_{B} satisfies the absolute purity property according to [CD12, Th. 14.4.1].
(2) According to the theorem of Gabber [Fuj02], the motivic category $\mathrm{D}^b_c(-,\Lambda)$ satisfies the absolute purity property for any quasi-excellent scheme, with Λ a finite ring of order prime to the residue characteristics of X.

A.3. **Torsion, homotopy and étale descent.** Recall the following result, essentially proved in [Voe96], but formulated in the premotivic triangulated category of Example A.1.9:

Proposition A.3.1. *For any scheme S of characteristic $p>0$, the category $\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Z})$ is $\mathbf{Z}[1/p]$-linear.*

Proof. The Artin-Schreier exact sequence ([AGV73, IX, 3.5]) can be written as an exact sequence of sheaves in $\mathrm{Sh}_{\text{ét}}(X,\mathbf{Z})$:

$$0 \to (\mathbf{Z}/p\mathbf{Z})_S \to \mathbf{G}_a \xrightarrow{F-1} \mathbf{G}_a \to 0$$

where F is the Frobenius morphism. But $\mathbf{G}_a$ is a strongly contractible sheaf, thus $F-1$ induces an isomorphism in the $\mathbf{A}^1$-localized derived category $\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Z})$. This implies $(\mathbf{Z}/p\mathbf{Z})_S=0$ in the latter category which in turn implies $p.Id$ is an isomorphism, as required. $\square$

A.3.2. Let $\mathscr{T}$ be a triangulated premotivic category. If $\mathscr{T}$ is obtained by a localization of the derived category of an abelian premotivic category, it comes with a canonical premotivic adjunction

$$D(\mathrm{PSh}(S,\mathbf{Z})) \rightleftarrows \mathscr{T}.$$

Then, the fact $\mathscr{T}$ satisfies the homotopy and the étale descent properties is equivalent to the fact that the previous adjunction induces a premotivic adjunction of the form:

$$\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}) \rightleftarrows \mathscr{T} \tag{A.3.2.a}$$

– see [CD12, 5.1.2, 5.2.10, 5.2.19, and 5.3.23].

Corollary A.3.3. *Let $\mathscr{T}$ be a premotivic triangulated category equipped with an adjunction of the form* (A.3.2.a)*. Then for any scheme S of characteristic $p>0$, $\mathscr{T}(S)$ is $\mathbf{Z}[1/p]$-linear.*

Proposition A.3.4. *Let p be a prime number and $n=p^a$ be a power of p. Let $\mathscr{T}$ be a premotivic triangulated category equipped with a premotivic adjunction of the form:*

$$t^*:\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z}) \rightleftarrows \mathscr{T}:t_*.$$

Let S be a scheme. We put $S[1/p]=S\times\mathrm{Spec}(\mathbf{Z}[1/p])$ and consider the canonical open immersion $j:S[1/p]\to S$. Then the functor

$$j^*:\mathscr{T}(S)\to\mathscr{T}(S[1/p])$$

is an equivalence of categories.

Proof. Note that the proposition is obvious when $\mathscr{T}=\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z})$ by the previous corollary and the localization property. In particular, for any object of the form $E=t^*(M)$ with M in $\mathrm{D}^{eff}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z})$, we have $j_\sharp j^*(E)\simeq E$. In particular, we have $j_\sharp j^*(\mathbb{1}_S)\simeq\mathbb{1}_S$. Therefore, for any object E of $\mathscr{T}(S)$, one has

$$j_\sharp j^*(E)\simeq j_\sharp(j^*(\mathbb{1}_S)\otimes E)\simeq j_\sharp j^*(\mathbb{1}_S)\otimes E\simeq \mathbb{1}_S\otimes E.$$

As the functor $j_\sharp$ is fully faithful, this readily implies the proposition. $\square$

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